C^*-ALGEBRAS GENERATED BY PARTIAL ISOMETRIES

ILWOO CHO AND PALLE JORGENSEN

Abstract. We prove a structure theorem for a finite set $\mathcal{G}$ of partial isometries in a fixed countably infinite dimensional complex Hilbert space H. Our result is stated in terms of the C^*-algebra generated by $\mathcal{G}$. The result is new even in the case of a single partial isometry which is not an isometry or a co-isometry; and in this case, it extends the Wold decomposition for isometries. We give applications to groupoid C^*-algebras generated by graph groupoids, and to partial isometries which have finite defect indices and which parametrize the extensions of a fixed Hermitian symetric operator with dense domain on the Hilbert space H. Our classification parameters for the "Wold decomposed" set $\mathcal{G}_W$ of our finite set $\mathcal{G}$ of partial isometries involve infinite and explicit Cartesian product sets, and they are computationally attractive. Moreover, our classification labels generalize the notion of defect indices in the special case of the family $\mathcal{G}_W$ of partial isometries from the Cayley transform theory and Hermitian extensions of unbounded Hermitian operators with dense domain.

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1. Introduction

In this paper, all Hilbert spaces are separable infinite dimensional. We will consider C^*-subalgebras of $B(H)$ generated by finitely many partial isometries on H, where $B(H)$ is the operator algebra consisting of all (bounded linear) operators. First, we characterize the C^*-subalgebras generated by a single partial isometry characterized by a $*$-isomorphism. We show that they are characterized by a $*$-isomorphism index in $(\mathbb{N}_0^\infty)^4$, where $\mathbb{N}_0^\infty = \mathbb{N} \cup \{0, \infty\}$, up to finite numbers. This shows that the C^*-subalgebras of $B(H)$ generated by a single partial isometry are characterized by quadruples of numbers in $\mathbb{N}_0^\infty$. More generally, we consider C^*-subalgebras of $B(H)$ generated by finitely many partial isometries. We prove that they are determined by certain combinatorial objects, called the conditional iterated glued graphs and their corresponding graph groupoids. By observing the block structures of the graph groupoids, we can have the block structures of a C^*-algebra generated by finitely many partial isometries in $B(H)$.

Let H be a Hilbert space and let $a \in B(H)$ be an operator, where $B(H)$ is the operator algebra consisting of all bounded linear operators on H. We say that this operator a is a *partial isometry* if the operator $a^*\,a$ is a projection on H, where a^* is the adjoint of a. Recall that an operator $p \in B(H)$ is a *projection* if p satisfies that $p^2 = p = p^*$. We study C^*-subalgebras generated by finitely many partial isometries in $B(H)$. We will extend the classification results of [13].

Directed graphs, both finite and infinite, play a role in a myriad of areas of applications; electrical network of resistors, the internet, and random walk in probability theory, to mention only a few. Some of the results in the subject stress combinatorial aspects of the graphs, while others have a more analytic slant. Applications to quantum theory fall in the latter group, and that is where Hilbert space and non-commuting operators play a role. To see this, consider a given graph G, and some matrix operation which transfers data encoded in one site (or vertex) of G onto another. If there is a notion of energy, or some other conserved quantity for the entire graph, then there is also a naturally associated Hilbert space, and with the energy now representing the norm-squared. Under these conditions, the matrix transfer between vertices will take the form of partial isometries: As a result, we will then be assigning partial isometries to the directed edges in G. In this paper we will derive some of the C^*-algebraic invariants associated with such an approach. Hence, the analytic approach asks for representations of the directed graphs, but each application motivates different representations.

Before turning to our main theorem, we recall some basic definitions and results in the literature which are natural precursors to our present model. A partial isometry a in a Hilbert space H is a linear mapping $a : H \rightarrow H$, which restricts to an isometry between two closed subspaces in H, the initial space H^a_{init} and the final space H^a_{fin}. In addition, it is assumed that a maps the orthogonal complement of H^a_{init} into $\{0\}$. The dimensions of the ortho-complements of the two spaces are called the defect indices of a. If the first defect index is 0, we say that a is an isometry, and if the second is 0, a co-isometry. While there are good classification theorems for isometries, e.g., [27]; the case of partial isometries when both the defect indices are non-zero have so far resisted classification. See however [28]. From the pioneering work of Akhiezer and Glazman [26], we know that the Cayley transform of unbounded densely defined skew-symmetric ordinary differential operators are partial isometries with finite defect indices. In Section 3.1, we generalize such indices.

The simplest and most primitive case, of course would be asking for a universal C^*-algebra which is generated by a single partial isometry. Using a certain grading on this algebra, the author of [33] constructs an associated analog of the Cuntz algebras [31]. Recall that the Cuntz algebra O_d is the C^*-algebra on d-isometries on H which have mutually orthogonal final spaces adding up to H.

A popular invariant is the KK-group, and with his universal algebra, Kandelaki gives a homotopical interpretation of KK-groups (See [33] and [37]). He proves that his universal C^*-algebra is homotopically equivalent to $M_2(\mathbb{C})$ up to stabilization by (2×2)-matrices. Therefore, these Kandelaki algebras are KK-isomorphic.

This paper provides not only a characterization of C^*-subalgebras of $B(H)$ generated by a single partial isometry (See Section 3.1) but also a topological reduced free block structures for the analysis of C^*-subalgebras of $B(H)$ generated by finitely many partial isometries on H. In particular, if a is a partial isometry, then we have the so-called Wold decomposition $u + s$ of a, where u is the unitary part of a and s is the shift part of a, and hence the C^*-subalgebra $C^*(\{a\})$ of $B(H)$ is the C^*-algebra $C^*(\{u, s\})$, which is $*$-isomorphic to $C^*(\{u\}) \oplus C^*(\{s\})$ inside $B(H)$, generated by u and s. We realize that $C^*(\{u\})$ and $C^*(\{s\})$ are characterized by certain graph groupoids $\mathbb{G}_u$ and $\mathbb{G}_s$, induced by the corresponding graphs G_u and G_s of u and s, respectively.

In conclusion, we show that if $\mathcal{G} = \{a_1, ..., a_N\}$ is a family of partial isometries in $B(H)$, and $\mathcal{G}_W = \{x_1, ..., x_n\}$, the corresponding Wold decomposed family of $\mathcal{G}$, and if $\mathcal{G}_G = \{G_{x_1}, ..., G_{x_n}\}$ is the family of corresponding graphs of $\mathcal{G}_W$, then the C^*-subalgebra $C^*(\mathcal{G})$ generated by $\mathcal{G}$ is $*$-isomorphic to the groupoid algebra $\mathcal{A} = C^*_\alpha(\mathbb{G})$, as embedded C^*-subalgebras of $B(H)$. Futhermore, the C^*-algebra $\mathcal{A}$ has its building blocks determined by the topological reduced free product.

Our motivations come from both pure and applied mathematics, including mathematical physics.

On the applied side, we notice that the internet offers graphs of very large size, hence to an approximation, infinite. This is a context where algebraic models have been useful (e.g., [55] and the references cited in it).

Features associated with infinite models are detected especially nicely with the geometric tools from operators on a Hilbert space. A case in point is the kind of Transfer Operator Theory or Spectral Theory which goes into the mathematics of internet search engines (e.g., [55]). A second instance is the use of graph models in the study of spin models in Quantum Statistical Mechanics (e.g., [30], [32], [49] and [50]). In both of these classes of "infinite" models mentioned above, what happens is there is an essential distinction between conclusions from the use of infinite dimensional models, as compared to the finite dimensional counterparts.

And there is the same distinction between the mathematics of a finite number of degrees of freedom vs. the statistics of infinite models: For a finite number of degrees of freedom, we have the Stone-von Neumann uniqueness theorem, but not so in the infinite case. Nonetheless, isomorphism classes of C^*-algebras are essential for the study of fields and statistics in Physics. In fact, these math physics models lie at the foundations of Operator Algebra Theory.

In Statistical Mechanics, we know that phase-transition may happen in certain infinite models, but of course is excluded for the corresponding finite models.

On the pure side, there are several as well: First, in the literature, there is a variety of classification theorems for C^*-algebras generated by systems of isometries, starting with the case of a single isometry; Coburn's Theorem (See [54]). Second, in Free Probability (e.g., [16] and [18]), C^*-algebras built on generators and relations are the context for "con-comutative" random variables.

2. Preliminaries

In this paper, we will consider a C^*-algebra $C^*(\mathcal{G})$ generated by a family $\mathcal{G}$ of finitely many partial isometries in $B(H)$. We first introduce the main objects we will need throughout the paper.

2.1. **Partial Isometries on a Hilbert Space.**

We say that an operator $a \in B(H)$ is a *partial isometry*, if the operators $a^*a \in B(H)$ is a projection. The various equivalent characterizations of partial isometries are well-known: the operator a is a partial isometry if and only if $a = aa^*a$, if and only if its adjoint a^* is a partial isometry, too, in $B(H)$. Recall that an operator p

in $B(H)$ is a projection if p is a self-adjoint idempotent. i.e., $p^2 = p = p^*$ in $B(H)$. Every partial isometry a has its initial space H_{init}^a and its final space H_{fin}^a which are closed subspaces of H. Notice that a partial isometry a is a unitary from H_{init}^a onto H_{fin}^a. i.e., $a \in B(H_{init}^a, H_{fin}^a)$ is a unitary satisfying that $a^*a = 1_{H_{init}^a}$ and $aa^* = 1_{H_{fin}^a}$, where 1_K means the identity operator on a Hilbert space K. Therefore, as Hilbert spaces, H_{init}^a and H_{fin}^a are Hilbert-space isomorphic.

2.2. **Directed Graphs and Graph Groupoids.**

Countable directed graphs and their applications have been studied extensively in Pure and Applied Mathematics (e.g., [4], [5], [6], [9], [10], [17], [19], [24], [35] and [36]). Not only are they connected with certain noncommutative structures but they also let us visualize such structures. Futhermore, the visualization has a nice matricial expressions.

A *graph* is a set of objects called vertices (or points or nodes) connected by links called edges (or lines). In a *directed graph*, the two directions are counted as being distinct directed edges (or arcs). A graph is depicted in a diagrammatic form as a set of dots (for vertices), jointed by curves (for edges). Similarly, a directed graph is depicted in a diagrammatic form as a set of dots jointed by arrowed curves, where the arrows point the direction of the directed edges.

Our global results and theorems have applications to Operator Theory as well as to a class of finite and infinite electrical networks used in Analysis on fractals (e.g., [53]), and Physics (e.g., [49] and [50]). We have given our results an axiometic formulation, making use here of a general Operator Theoretic framework. The operator-theoretic approach is new, especially in its general scope. When stated in the context of Hilbert-space operators, we believe that our results are also of independent interest; quite apart from their other applications. Our results have a variety of applications, e.g., to fractals with affine selfsimilarity (See [53]), to models in Quantum Statistical Mechanics (See [51]), to the analysis of energy forms, interplaying between selfsimilar measures and associated energy forms; and to random walks on graphs (See [53]).

By a graph G, we will mean a system of vertices $V(G)$ and edges $E(G)$ (See the previous axioms below). Intuitively, the points in $E(G)$ will be "lines" connecting prescribed pairs of points in $V(G)$. Each vertex x will have a finite set $g(x)$ of edges connecting it to other points in $V(G)$. (For models in Physics, each set $g(x)$ will correspond to a set of nearest neighbors for the vertex x, with the vertices typically arranged in a lattice (See [49], [50] and [51]).)

But our present graphs will be more general. Starting with G, we specify a function g from $V(G)$ into the set of all finite subsets of $V(G)$, subject to certain axioms; see below for details.

An important difference between there graphs and those that are more traditional in Geometric Analysis is that we do not "a priori" fix an orientation. Nonetheless, for electrical network models, we will be using orientations, but they will be introduced only indirectly, via a prescribed function R, positive, real valued and defined on $E(G)$, i.e., on the set of edges on G. (The letter R is for resistance.) By analogy to models in Electrical Engineering, we think of such a function R as a system of resistors, and we will be interested in the pair (G, R). Given a system R of resistors, then potential theoretic problems for (G, R) will induce currents in the global system, the associated current functions will be denoted by I, and each admissible (subject to Kirchhoff's laws) current function induces an orientation.

This is a second class of functions I (I, for electrical current) again functions from $E(G)$ into the real numbers. It is only when such a function I is prescribed, that our graph G will acquire an orientation: Specially we say that an edge e is positively oriented (relative to I), if $I(e) > 0$. And a different choice of current function I will induce a different orientation on the graph G.

Our results will be stated in terms of precise mathematical axioms, and we present our theorems in the framework of Operator Theory / Operator Algebra, which happens to have an element of motivation from the theory of infinite systems of resistors. The main tools in our proofs will be operators on a Hilbert space. In particular, we will solve problems in Discrete Potential Theory with the use of a family of operators and their adjoints. Good references to the fundamentals for operators are [44] and [48]. The second author of this paper recalls conversations with Raul Bott with a remainder about an analogous Hilbert-space Operator Theoretic approach to Electrical Networks; apparently attempted in the 1950's in the engineering literature, but we did not find details in journals. The closest we could come is the fascinating paper [45], by Bott et al.

A motivation for our present study is a series of papers in the 1970's by Powers which introduced infinite systems of resistors into the resolution of an important question from Quantum Statistical Mechanics (e.g., [49], [50], [51] and [52]). This is coupled with the emergence of these same techniques, or their close vintage. By "Analysis on Fractals" we refer to a set of Potential Theoretic tools developed recently by Kigami, Strichartz and others (e.g., [44] and [46]).

In this paper, we observe how an algebraic structure, a graph groupoid induced by a finite directed graph, acts on a Hilbert space H. In [9] and [10], we showed that every graph groupoid maybe realized with a system of Hilbert space operators on a so-called graph Hilbert space.

Let G be a countable directed graph with its vertex set $V(G)$ and its edge set $E(G)$. Denote the set of all finite paths of G by $FP(G)$. Clearly, the edge set $E(G)$ is contained in $FP(G)$. Let w be a finite path in $FP(G)$. Then it is represented as a word in $E(G)$. If e_1, ..., e_n are connected directed edges in the order $e_1 \to e_2 \to ... \to e_n$, for $n \in \mathbb{N}$, then we can express w by e_1 ... e_n in $FP(G)$. If there exists a finite path $w = e_1 \ ... \ e_n$ in $FP(G)$, where $n \in \mathbb{N} \setminus \{1\}$, we say that the directed edges e_1, ..., e_n are admissible. The length $|w|$ of w is defined to be n, which is the

cardinality of the admissible edges generating w. Also, we say that finite paths $w_1 = e_{11} \ldots e_{1k_1}$ and $w_2 = e_{21} \ldots e_{2k_2}$ are admissible, if $w_1\ w_2 = e_{11} \ldots e_{1k_1}\ e_{21} \ldots e_{2k_2}$ is again in $FP(G)$, where $e_{11}, \ldots, e_{1k_1}, e_{21}, \ldots, e_{2k_2} \in E(G)$. Otherwise, we say that w_1 and w_2 are not admissible. Suppose that w is a finite path in $FP(G)$ with its initial vertex v_1 and its terminal vertex v_2. Then we write $w = v_1 w$ or $w = w v_2$ or $w = v_1 w v_2$, for emphasizing the initial vertex of w, respectively the terminal vertex of w, respectively both the initial vertex and the terminal vertex of w. Suppose $w = v_1 w v_2$ in $FP(G)$ with $v_1,\ v_2 \in V(G)$. Then we also say that [v_1 and w are admissible] and [w and v_2 are admissible]. Notice that even though the elements w_1 and w_2 in $V(G) \cup FP(G)$ are admissible, w_2 and w_1 are not admissible, in general. For instance, if $e_1 = v_1 e_1 v_2$ is an edge with $v_1,\ v_2 \in V(G)$ and $e_2 = v_2 e_2 v_3$ is an edge with $v_3 \in V(G)$ such that $v_3 \neq v_1$, then there is a finite path $e_1\ e_2$ in $FP(G)$, but there is no finite path $e_2\ e_1$.

The *free semigroupoid* $\mathbb{F}^+(G)$ of G is defined by a set

$$\mathbb{F}^+(G) = \{\emptyset\} \cup V(G) \cup FP(G),$$

with its binary operation $(\cdot)$ on $\mathbb{F}^+(G)$, defined by

$$(w_1, w_2) \mapsto w_1 \cdot w_2 = \begin{cases} w_1 & \text{if } w_1 = w_2 \text{ in } V(G) \\ w_1 & \text{if } w_1 \in FP(G),\ w_2 \in V(G) \text{ and } w_1 = w_1 w_2 \\ w_2 & \text{if } w_1 \in V(G),\ w_2 \in FP(G) \text{ and } w_2 = w_1 w_2 \\ w_1 w_2 & \text{if } w_1,\ w_2 \text{ in } FP(G) \text{ and } w_1 w_2 \in FP(G) \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\emptyset$ is the empty word in $V(G) \cup E(G)$. (Sometimes, the free semigroupoid $\mathbb{F}^+(G)$ of a certain graph G does not contain the empty word $\emptyset$. For instance, the free semigroupoid of the one-vertex–multi-loop-edge graph does not have the empty word. But, in general, the empty word $\emptyset$ is contained in the free semigroupoid, whenever $|V(G)| > 1$. So, if there is no confusion, then we usually assume that the empty word is contained in free semigroupoids.) This binary operation $(\cdot)$ on $\mathbb{F}^+(G)$ is called the *admissibility.* i.e., the algebraic structure $(\mathbb{F}^+(G), \cdot)$ is the free semigroupoid of G. For convenience, we denote $(\mathbb{F}^+(G), \cdot)$ simply by $\mathbb{F}^+(G)$.

For the given countable directed graph G, we can define a new countable directed graph G^{-1} which is the opposite directed graph of G, with

$$V(G^{-1}) = V(G) \quad \text{and} \quad E(G^{-1}) = \{e^{-1} : e \in E(G)\},$$

where $e^{-1} \in E(G^{-1})$ is the opposite directed edge of $e \in E(G)$, called the *shadow* of $e \in E(G)$. i.e., if $e = v_1\ e\ v_2$ in $E(G)$ with $v_1,\ v_2 \in V(G)$, then $e^{-1} = v_2\ e^{-1}\ v_1$ in $E(G^{-1})$ with $v_2,\ v_1 \in V(G^{-1}) = V(G)$. This new directed graph G^{-1} is said to be *the shadow of* G. It is trivial that $(G^{-1})^{-1} = G$. This relation shows that the admissibility on the shadow G^{-1} is oppositely preserved by that on G.

A new countable directed graph $\widehat{G}$ is called the *shadowed graph of* G if it is a directed graph with

$$V(\widehat{G}) = V(G) = V(G^{-1})$$

and

$$E(\widehat{G}) = E(G) \cup E(G^{-1}).$$

Definition 2.1. *Let* G *be a countable directed graph and* $\widehat{G}$, *the shadowed graph of* G, *and let* $\mathbb{F}^{+}(\widehat{G})$ *be the free semigroupoid of* $\widehat{G}$. *Define the reduction (RR) on* $\mathbb{F}^{+}(\widehat{G})$ *by*

$$\text{(RR)} \qquad ww^{-1} = v \quad \text{and} \quad w^{-1}w = v',$$

whenever $w = vwv'$ *in* $FP(\widehat{G})$, *with* $v,\ v' \in V(\widehat{G})$. *The set* $\mathbb{F}^{+}(\widehat{G})$ *with this reduction (RR) is denoted by* $\mathbb{F}_{r}^{+}(\widehat{G})$. *And this set* $\mathbb{F}_{r}^{+}(\widehat{G})$ *with the inherited admissibility* $(\cdot)$ *from* $\mathbb{F}^{+}(\widehat{G})$ *is called the graph groupoid of* G. *Denote* $(\mathbb{F}_{r}^{+}(\widehat{G}),\ \cdot)$ *of* G *by* $\mathbb{G}$. *Define the reduced finite path set* $FP_{r}(\widehat{G})$ *of* $\mathbb{G}$ *by*

$$FP_{r}(\widehat{G}) \overset{def}{=} \mathbb{G} \setminus (V(\widehat{G}) \cup \{\emptyset\}).$$

All elements of $FP_{r}(\widehat{G})$ *are said to be reduced finite paths of* $\widehat{G}$.

Let w_1 and w_2 be reduced finite paths in $FP_r(\widehat{G}) \subset \mathbb{G}$. We will use the same notation $w_1\ w_2$ for the (reduced) product of w_1 and w_2 in $\mathbb{G}$. But we have to keep in mind that the product $w_1\ w_2$ in the graph groupoid $\mathbb{G}$ is different from the (non-reduced) product $w_1\ w_2$ in the free semigroupoid $\mathbb{F}^{+}(\widehat{G})$. Suppose e_1 and e_2 are edges in $E(\widehat{G})$ and assume that they are admissible, and hence $e_1\ e_2$ is a finite path in $FP(\widehat{G})$. Then the product $e_1\ e_2\ e_2^{-1}$ of $e_1\ e_2$ and e_2^{-1} is a length-3 finite path in $FP(\widehat{G}) \subset \mathbb{F}^{+}(\widehat{G})$, but the product $e_1\ e_2\ e_2^{-1}$ of them is $e_1\ (e_2\ e_2^{-1}) = e_1$, which is a length-1 reduced finite path, in $FP_r(\widehat{G}) \subset \mathbb{G}$.

2.3. C^{*}-Algebras Induced by Graphs.

In [13], we observed the C^{*}- algebras generated by certain family of partial isometries on a fixed Hilbert space H. Suppose a family $\mathcal{G} = \{a_1,\ ...,\ a_N\}$ in $B(H)$ is a collection of finite number of partial isometries, where $N \in \mathbb{N}$. We say that such family $\mathcal{G}$ *construct a finite directed graph* G if there exists a finite directed graph G such that

(i) $|E(G)| = |\mathcal{G}|$, and $|V(G)| = |\mathcal{G}_{pro}|$, where

$$\mathcal{G}_{pro} = \{a^* \ a : a \in \mathcal{G}\} \cup \{a \ a^* : a \in \mathcal{G}\},$$

(ii) the edges e_1 and e_2 create a nonempty finite path $e_1 \ e_2$ on G if and only if $H_{init}^{a_1} = H_{fin}^{a_2}$, where "$=$" means "being identically same in H".

In this case, the family $\mathcal{G}$ is called a *G-family of partial isometries* in $B(H)$. The family $\widehat{\mathcal{G}} = \mathcal{G} \cup \mathcal{G}^*$ generates a C^*-subalgebra $C^*(\mathcal{G})$ of $B(H)$, where $\mathcal{G}^* = \{a_j^* : j = 1, ..., N\}$.

Let $\mathbb{G}$ be the graph groupoid of a finite directed graph G, with $\left|V(\widehat{G})\right| = n$ and $\left|E(\widehat{G})\right| = 2N$ (i.e., $|E(G)| = N$). We give a indexing on $V(G)$ by $\{1, ..., n\}$. i.e., we will let $V(G) = \{v_1, ..., v_n\}$. By doing that, we can also index the elements $E(G)$ and $E(\widehat{G})$ as follows:

$$E(G) = \left\{ e_{m:ij} \left| \begin{array}{c} m = 1, \ ..., k_{ij}, \ k_{ij} \neq 0 \\ e_{m:ij} = v_i e_{m:ij} v_j \end{array} \right. \right\}$$

and

$$E(\widehat{G}) = \left\{ x_{m:ij} \left| \begin{array}{c} x_{m:ij} = e_{m:ij} \text{ if } x_{m:ij} \in E(G) \\ x_{m:ij} = e_{m:ji}^{-1} \text{ if } x_{m:ij} \in E(G^{-1}) \\ m = 1, ..., k_{ij}, \ k_{ij} \neq 0 \end{array} \right. \right\},$$

where k_{ij} means the cardinality of edges connecting the vertex v_i to v_j. By the finiteness of G, $k_{ij} < \infty$, whenever $k_{ij} \neq 0$. Clearly, "$k_{ij} = 0$" means that "there is no edge connecting v_i to v_j". In [13], we showed that such indexing process on $V(G) \cup E(G)$ is uniquely determined up to graph-isomorphisms. This means that if we fix an indexing, then this indexing contains the full (admissibility) combinatorial data of G.

We will say that a finite directed graph G is *connected*, if its graph groupoid $\mathbb{G}$ satisfies that: for any $(v_1, v_2) \in V(G) \times V(G)$, where $v_1 \neq v_2$, there exists an element $w \in \mathbb{G}$ such that $w = v_1 \ w \ v_2$ and $w^{-1} = v_2 \ w^{-1} \ v_1$. Otherwise, we say that the graph G is not connected or is disconnected.

Under this setting, if G is a connected finite directed graph, then the graph groupoid $\mathbb{G}$ has its matricial representation $(\mathcal{H}_G, \pi)$, where $\mathcal{H}_G$ is a Hilbert space defined by $\oplus_{j=1}^n (\mathbb{C} \ \xi_{v_j})$, which is Hilbert-space isomorphic to $\mathbb{C}^{\oplus n}$, and where $\pi : \mathbb{G} \to B(\mathcal{H}_G)$ is a groupoid action satisfying that

$$\pi(v_j) = P_j$$

and

$$\pi(x_{m:ij}) = \begin{cases} E_{m:ij} & \text{if } x_{m:ij} = e_{m:ij} \\ \\ E_{m:ji}^* & \text{if } x_{m:ij} = e_{m:ji}^{-1} \ , \end{cases}$$

where P_j is the diagonal matrix in $M_n(\mathbb{C})$, having its only nonzero $(j,\, j)$-entry 1,

$$\begin{array}{cc} \begin{matrix} & & j\text{-th} & & \end{matrix} & \\ \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} & j\text{-th} \end{array}$$

and $E_{m:ij}$ is the matrix in $M_n(\mathbb{C})$, having its only nonzero $(i,\, j)$-entry ω^m, where ω is the root of unity of the polynomial $z^{k_{ij}}$, whenever $i \neq j$,

$$\begin{array}{cc} \begin{matrix} & i\text{-th} & \end{matrix} & \\ \begin{pmatrix} 0 & & 0 \\ & \omega^m & \\ 0 & & 0 \end{pmatrix} & j\text{-th} \end{array}$$

or it is the diagonal matrix in $M_n(\mathbb{C})$, having its only nonzero $(j,\, j)$-entry $e^{i\,\theta_{m:jj}}$, where $\theta_{m:jj} \in \mathbb{R} \setminus \{0\}$, satisfying that $\theta_{m_1:jj} \neq \theta_{m_2:jj}$, whenever $m_1 \neq m_2$.

$$\begin{array}{cc} \begin{matrix} & & j\text{-th} & & \end{matrix} & \\ \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & e^{i\theta_{jj}} & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} & j\text{-th} \end{array}$$

Then we can see that the graph groupoid $\mathbb{G}$ generates the C^*- algebra $\mathcal{M}_G$ which is a C^*-subalgebra of $B(\mathcal{H}_G) \overset{*\text{-isomorphic}}{=} M_n(\mathbb{C})$. This C^*-subalgebra $\mathcal{M}_G$ induced by G in $M_n(\mathbb{C})$ is called the *matricial graph C^*-algebra* of a connected finite directed graph G.

If the family $\mathcal{G}$ of partial isometries in $B(H)$ constructs a connected finite directed graph G, then, since $\mathcal{G}$ generates a groupoid $\mathbb{G}_{\mathcal{G}}$, under the operator multiplication on $B(H)$, and it is groupoid-isomorphic to a certain graph groupoid $\mathbb{G}$, we can check that the C^*-subalgebra $C^*(\mathcal{G})$ of $B(H)$ is $*$-isomorphic to the affiliated matricial graph C^*- algebra

$$\mathcal{M}_G(H_0) \overset{def}{=} (\mathbb{G} \cdot 1_{H_0}) \otimes_{\mathbb{C}} \mathcal{M}_G \overset{C^*\text{-subalgebra}}{\subseteq} M_n(B(H_0)),$$

for certain Hilbert space H_0 which is embedded in H, where

$$M_n\left(B(H_0)\right) = \left\{ \left(\begin{array}{ccc} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{array} \right) \middle| T_{ij} \in B(H_0) \right\}.$$

We conclude this section with two results which will be extended later.

Theorem 2.1. *(See [13]) Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$, and assume that $\mathcal{G}$ constructs a connected finite directed graph G. Then the C^*-subalgebra $C^*(\mathcal{G})$ of $B(H)$ is $*$-isomorphic to the affiliated matricial algebra $\mathcal{M}_G(H_0)$, where H_0 is a Hilbert space which is Hilbert-space isomorphic to $p\ H$, for all $p \in \mathcal{G}_{pro}$.* $\square$

Corollary 2.2. *(Also See [13]) Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$, and assume that $\mathcal{G}$ constructs a finite directed graph G, having its connected components G_1, ..., G_t, for $t \in \mathbb{N}$. Then the C^*-subalgebra $C^*(\mathcal{G})$ of $B(H)$ is $*$-isomorphic to the direct product algebra $\oplus_{j=1}^{t}\ \mathcal{M}_G(H_j)$ of the affiliated matricial graph C^*-algebras $\mathcal{M}_{G_j}(H_j) \overset{def}{=} (\mathbb{C} \cdot 1_{H_j}) \otimes_{\mathbb{C}} \mathcal{M}_{G_j}$, where $\mathcal{M}_{G_j}$'s are the matricial C^*-algebras induced by G_j, for $j = 1$, ..., N.* $\square$

The above theorem and corollary are the main motivation of this paper. They provide a certain connection between partial isometries and graph groupoids. We will extend the above results. We realize that, in general, we have much more complicated structures than the above special cases.

2.4. **Groupoids and Groupoid Actions.**

As we observed in [9], [10] and [12], every graph groupoid is indeed a (categorial) groupoid. This means that graph groupoids have rough but rich algebraic structures. The following definition is inspired by [19] and [25].

Definition 2.2. *We say an algebraic structure $(\mathcal{X},\ \mathcal{Y},\ s,\ r)$ is a (categorial) groupoid if it satisfies that (i) $\mathcal{Y} \subset \mathcal{X}$, (ii) for all $x_1, x_2 \in \mathcal{X}$, there exists a partially-defined binary operation $(x_1,\ x_2) \mapsto x_1\ x_2$, for all $x_1,\ x_2 \in \mathcal{X}$, depending on the source map s and the range map r satisfying that:*

(ii-1) the operation $x_1\ x_2$ is well-determined, whenever $r(x_1)\ s(x_2) \in \mathcal{Y}$,

(ii-2) the operation $(x_1\ x_2)\ x_3 = x_1\ (x_2\ x_3)$ is defined, if the constituents are well-determined in the sense of (ii-1), for $x_1,\ x_2,\ x_3 \in \mathcal{X}$,

(ii-3) if $x \in \mathcal{X}$, then there exist y, $y' \in \mathcal{Y}$ such that $s(x) = y$ and $r(x) = y'$, satisfying $x = y\ x\ y'$,

(ii-4) if $x \in \mathcal{X}$, then there exists a unique groupoid-inverse x^{-1} satisfying $x\ x^{-1} = s(x)$ and $x^{-1}\ x = r(x)$, in $\mathcal{Y}$.

By definition, we can conclude that every group is a groupoid ($\mathcal{X}$, $\mathcal{Y}$, s, r) with $|\mathcal{Y}| = 1$ (and hence $s = r$ on $\mathcal{X}$). This subset $\mathcal{Y}$ of $\mathcal{X}$ is said to be the base of $\mathcal{X}$. Remark that we can naturally assume that there exists the empty element $\emptyset$ in a groupoid $\mathcal{X}$. Roughly speaking, the empty element $\emptyset$ means that there are products $x_1\ x_2$ which are not well-defined, for x_1, $x_2 \in \mathcal{X}$. Notice that if $|\mathcal{Y}| = 1$ (equivalently, if $\mathcal{X}$ is a group), then the empty word $\emptyset$ is not contained in the groupoid $\mathcal{X}$. However, in general, whenever $|\mathcal{Y}| \geq 2$, a groupoid $\mathcal{X}$ always contain the empty word. So, if there is no confusion, we will automatically assume that the empty element $\emptyset$ is contained in $\mathcal{X}$.

It is easily checked that our graph groupoid $\mathbb{G}$ of a finite directed graph G is indeed a groupoid with its base $V(\widehat{G})$. i.e., every graph groupoid $\mathbb{G}$ of a countable directed graph G is a groupoid ($\mathbb{G}$, $V(\widehat{G})$, s, r), where $s(w) = s(v\ w) = v$ and $r(w) = r(w\ v') = v'$, for all $w = v\ w\ v' \in \mathbb{G}$ with v, $v' \in V(\widehat{G})$.

Let $\mathcal{X}_k = (\mathcal{X}_k, \mathcal{Y}_k, s_k, r_k)$ be groupoids, for $k = 1, 2$. We say that a map $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ is a *groupoid morphism* if (i) $f(\mathcal{Y}_1) \subseteq \mathcal{Y}_2$, (ii) $s_2(f(x)) = f(s_1(x))$ in $\mathcal{X}_2$, for all $x \in \mathcal{X}_1$, and (iii) $r_2(f(x)) = f(r_1(x))$ in $\mathcal{X}_2$, for all $x \in \mathcal{X}_1$. If a groupoid morphism f is bijective, then we say that f is a *groupoid-isomorphism*, and that the groupoids $\mathcal{X}_1$ and $\mathcal{X}_2$ are *groupoid-isomorphic*.

Notice that, if two countable directed graphs G_1 and G_2 are graph-isomorphic, via a graph-isomorphism $g : G_1 \rightarrow G_2$, in the sense that (i) g is bijective from $V(G_1)$ onto $V(G_2)$, (ii) g is bijective from $E(G_1)$ onto $E(G_2)$, (iii) $g(e) = g(v_1\ e\ v_2) = g(v_1)\ g(e)\ g(v_2)$ in $E(G_2)$, for all $e = v_1\ e\ v_2 \in E(G_1)$, with v_1, $v_2 \in V(G_1)$, then the graph groupoids $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomorphic. More generally, if G_1 and G_2 have graph-isomorphic shadowed graphs $\widehat{G}$ and $\widehat{G_2}$, then $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomorphic (See Section 3, in detail).

Let $\mathcal{X} = (\mathcal{X}, \mathcal{Y}, s, r)$ be a groupoid. We say that this groupoid $\mathcal{X}$ acts on a set X if there exists a groupoid action π of $\mathcal{X}$ such that $\pi(x) : X \rightarrow X$ is a well-determined function. Sometimes, we call the set X, a $\mathcal{X}$-set. We are interested in the case where a $\mathcal{X}$-set X is a Hilbert space. A nice example of groupoid actions acting on a Hilbert space are graph-representations defined and observed in [9]. In this paper, we will consider a new kind of graph-groupoid action on certain Hilbert spaces.

3. C^*-Subalgebras of $B(H)$ Generated by Partial Isometries

The section has two parts, one centered around our Theorem 3.3. This result generalizes the more traditional notion of deficiency indices (See Section 5). The second part shows that our C^*-algebras can be realized in representations of groupoids.

In [13], we considered the case where the collection of finite partial isometries are graph-families. How about the general cases where the family of finite partial isometries on H is not a graph-family in the sense of [13]? We provide an answer of this question. As before, we will fix a Hilbert space H and the corresponding operator algebra $B(H)$. Let $\mathcal{G} = \{a_1, ..., a_N\}$ be a family of partial isometries $a_1, ..., a_N$ of $B(H)$, where $N \in \mathbb{N}$. Compared with [13], we will have much more complicated results.

3.1. $*$-Isomorphic Indices of Partial Isometries.

In this section, we will classify the set of all partial isometries of $B(H)$. For convenience, we denote the subset of all partial isometries of $B(H)$ by $PI(H)$. Let $a \in PI(H)$ with its initial space H_{init}^a and its final space H_{fin}^a. Recall that $H_{init}^a = (a^* \, a) \, H$ and $H_{fin}^a = (a \, a^*) \, H$. Then we can find the subspaces

$$(H_{init}^a)^\perp = H \ominus H_{init}^a = (1_H - a^* a) \, H = \ker a$$

and

$$(H_{fin}^a)^\perp = H \ominus H_{fin}^a = (1_H - a \, a^*) \, H = \ker a^*.$$

Assume that

$$\varepsilon_+ \overset{def}{=} \dim \left((H_{init}^a)^\perp\right) \text{ and } \varepsilon_- \overset{def}{=} \dim \left((H_{fin}^a)^\perp\right),$$

where $\varepsilon_+, \varepsilon_- \in \mathbb{N}_0^\infty = \mathbb{N} \cup \{0, \infty\}$. Then we can determine a subset $PI_H(\varepsilon_+, \varepsilon_-)$ of $PI(H)$ by

$$PI_H(\varepsilon_+, \varepsilon_-) \overset{def}{=} \left\{ a \in PI(H) \,\middle|\, \begin{array}{l} \dim\left((H_{init}^a)^\perp\right) = \varepsilon_+, \\ \dim\left((H_{fin}^a)^\perp\right) = \varepsilon_- \end{array} \right\}.$$

Note that

$$PI(H) = \underset{(\varepsilon_+, \, \varepsilon_-) \in (\mathbb{N}_0^\infty)^2}{\cup} \left(PI_H(\varepsilon_+, \, \varepsilon_-)\right),$$

set-theoretically, in $B(H)$, where $\mathbb{N}_0^\infty = \mathbb{N} \cup \{0, \infty\}$. For instance, if U is the well-known unilateral shift (which is an isometry) on the Hilbert space $l^2(\mathbb{N}_0)$, sending $(\xi_0, \xi_1, \xi_2, ...) \in l^2(\mathbb{N}_0)$ to

$$U\left((\xi_0,\ \xi_1,\ \xi_2,\ ...)\right) \overset{def}{=} (0, \xi_0, \xi_1, \xi_2, ...) \in l^2(\mathbb{N}_0),$$

where $\mathbb{N}_0 \overset{def}{=} \mathbb{N} \cup \{0\}$, then it is contained in $PI_{l^2(\mathbb{N}_0)}(0, 1)$, since $H_{init}^U = l^2(\mathbb{N}_0)$ and $H_{fin}^U = l^2(\mathbb{N}_0) \ominus \mathbb{C}$. Notice that every operator U^ε on $l^2(\mathbb{N}_0)$ can be regarded as elements in $PI_{l^2(\mathbb{N}_0)}(0, \varepsilon)$, for all $\varepsilon \in \mathbb{N}$, and every unitaries on an arbitrary Hilbert space K can be regarded as elements in $PI_K(0, 0)$.

Let $u_1 \neq u_2 \in PI_H(0, 0)$ be unitaries, satisfying that $spec(u_1) = spec(u_2)$ in $\mathbb{T} \subset \mathbb{C}$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$ and $spec(x)$ means the spectrum of x, for all $x \in B(H)$. Then we can get that:

Lemma 3.1. *If the subspaces H_{u_1} and H_{u_2} of H are Hilbert-space isomorphic, where $H_{u_k} = (u_k^* \ u_k)\ H$, for $k = 1, 2$, and if the spectrums $spec(u_1)$ and $spec(u_2)$ are identical in $\mathbb{T} \subset \mathbb{C}$, then the C^*-subalgebras $C^*(\{u_1\})$ and $C^*(\{u_2\})$ of $B(H)$ are $*$-isomorphic.*

Proof. By [13], we have that the C^*-algebras $C^*(\{u_k\})$ are $*$-isomorphic to $(\mathbb{C} \cdot 1_{H_{u_k}}) \otimes_{\mathbb{C}} C\left(spec(u_k)\right)$, as embedded C^*-subalgebra of $B(H)$, for $k = 1, 2$, where $C(X)$ means the C^*-algebra consisting of all continuous functions on a compact subset X of $\mathbb{C}$. Since the subspaces H_{u_1} and H_{u_2} are Hilbert-space isomorphic in H and since the spectrums $spec(u_1)$ and $spec(u_2)$ are same in $\mathbb{T}$, the C^*-algebras $(\mathbb{C} \cdot 1_{H_k}) \otimes_{\mathbb{C}} C\left(spec(u_k)\right)$'s are $*$-isomorphic C^*-subalgebras of $B(H)$. ∎

If $a \in PI(H)$ is a partial isometry on H, then we can regard a as an isometry on $H \ominus \ker a$. Recall that if $x \in B(K)$ is an isometry on a Hilbert space K, then x has its Wold decomposition $x = u_x + s_x$, where u_x is the unitary part of x and s_x is the shift part of x. Futhermore, the Hilbert space K is decomposed by $K = K_{u_x} \oplus K_{s_x}$, where K_{u_x} (resp., K_{s_x}) is the subspace of K where u_x (resp., s_x) is acting on, as a unitary (resp., as a shift).

Definition 3.1. *Suppose $a \in PI(H)$ is a partial isometry. We say that $u_a + s_s \in B(H)$ is the Wold decomposition of a, if u_a is the unitary part of a on H_a and s_a is the shift part of a on H_a, by regarding a as an isometry on H_a, where $H_a = (a^*\ a)\ H = H \ominus \ker a$. Equivalently, the Wold decomposition of $a \mid_{H_a}$ of the isometry $a \mid_{H_a}$ is said to be the Wold decomposition of the partial isometry a.*

Let $a \in PI_H(\varepsilon_+, \varepsilon_-)$. Then, by the Wold decomposition, we can decompose a

$$a = u_a + s_a,$$

where u_a is the unitary part on $H_{u_a} = (u_a^* \, u_a)\, H$ of a; and s_a is the shift part on $H_{s_a} = (s_a^* \, s_a)\, H$ of a. Note that the subspaces H_{u_a} and H_{s_a} of H are also the subspaces of the initial space H_{init}^a of a, and they decompose H_{init}^a. i.e., $H_{init}^a = H_{u_a} \oplus H_{s_a}$. We will call the subspaces H_{u_a} and H_{s_a} of H, the *unitary part of* H and the *shift part of* H, *in terms of* a.

By the Wold decomposition of a, we then get the following diagram:

$$\begin{array}{ccccccc} H & = & H_{u_a} & \oplus & H_{s_a} & \oplus & \ker a \\ \downarrow_a & & \downarrow_{u_a} & & \downarrow_{s_a} & & \\ H & = & H_{u_a} & \oplus & (\ker s_a^* \oplus H_{s_a}) & \oplus & \ker a^* \ominus \ker s_a^* \end{array}$$

i.e., the partial isometry a has the following block operator-matricial form,

$$a = \begin{pmatrix} u_a & & \\ & s_a & \\ & & 0 \end{pmatrix},$$

on $H_{u_a} \oplus H_{s_a} \oplus \ker a$.

Let $a \in PI_H(\varepsilon_+, \, \varepsilon_-)$, with its Wold decomposition $a = u_a + s_a$, and assume that the shift part s_a is contained in $PI_{H_{s_a}}(0, \, \varepsilon^-)$. Then we have that

$$\varepsilon_- = \varepsilon^- + \varepsilon_-^-, \text{ for some } \varepsilon_-^- \in \mathbb{N}_0^\infty,$$

where

$$\varepsilon^- = \dim (\ker s_a^*) \text{ and } \varepsilon_-^- = \varepsilon_- - \varepsilon^-.$$

Definition 3.2. *Let $a \in PI_H(\varepsilon_+, \, \varepsilon_-)$ with its Wold decomposition $u_a + s_a$, and assume that $s_a \in PI_{H_{s_a}}(0, \, \varepsilon^-)$. Then we will denote the collection of such partial isometries a by $PI_H(\varepsilon_+, \, \varepsilon^-, \, \varepsilon_-^-)$.*

Let $a \in PI_H(\varepsilon_+, \, \varepsilon^-, \, \varepsilon_-^-)$ be a partial isometry. Then we can have the following properties.

Proposition 3.2. *Let $a \in PI_H(\varepsilon_+, \, \varepsilon^-, \, \varepsilon_-^-)$ be a nonzero partial isometry with its Wold decomposition $a = u + s$, where u and s are the unitary part and the shift part of a, respectively.*

(1) We can have that

$$C^*(\{a\}) \overset{*\text{-isomorphic}}{=} C^*(\{u, \, s\}) = C^*(\{u\}) \oplus C^*(\{s\}) \overset{C^*\text{-subalgebra}}{\subseteq} B(H).$$

(2) If $s = 0$, then

$$C^*(\{a\}) = C^*(\{u\}) \overset{*\text{-}isomorphic}{=} (\mathbb{C} \cdot 1_{H_u}) \otimes_{\mathbb{C}} C\,(spec(u))\,.$$

(3) If $u = 0$ and if $s \in PI_{H_s}(0,\, \varepsilon^-)$ with $\varepsilon^- \in \mathbb{N}$, then

$$C^*(\{a\}) = C^*(\{s\}) \overset{*\text{-}isomorphic}{=} \mathcal{T}(H_s),$$

where $\mathcal{T}(H_s)$ means the classical Toeplitz algebra defined on H_s.

(4) If $u = 0$ and if $s \in PI_{H_s}(0,\, \infty)$, then

$$C^*(\{a\}) = C^*(\{s\}) \overset{*\text{-}isomorphic}{=} (\mathbb{C} \cdot 1_{H_s}) \otimes_{\mathbb{C}} M_2(\mathbb{C}).$$

Proof. (1) Let $a \in PI_H(\varepsilon_+,\, \varepsilon^-,\, \varepsilon_-^-)$ be a partial isometry on H having its Wold decomposition $a = u + s$, where u is the unitary part of a on H_u and s is the shift part of a on H_s, where the subspaces H_u and H_s are the unitary part and the shift part of H in terms of a. Recall the diagram

$$\begin{array}{ccccccc} H & = & H_u & \oplus & H_s & \oplus & \ker a \\ \downarrow_a & & \downarrow_u & & \downarrow_s & & \\ H & = & H_u & \oplus & \ker s^* \oplus H_s & \oplus & (\ker a^* \ominus \ker s^*). \end{array}$$

Notice that $\ker s^* \oplus H_s$ is Hilbert-space isomorphic to H_s, since s is a shift. So, the C^*-subalgebra $C^*(\{a\})$ generated by a is $*$-isomorphic to the C^*-algebra $C^*(\{u,\, s\})$ generated by u and s. Moreover, since H_u and H_s are orthogonal (as subspaces) in H, the C^*-subalgebra $C^*(\{u,\, s\})$ is $*$-isomorphic to $C^*(\{u\}) \oplus C^*(\{s\})$ as an embedded C^*-subalgebra of $B(H)$.

(2) Suppose the shift part s of a partial isometry a is the zero operator. Then $a = u$, and hence the C^*-subalgebra $C^*(\{a\})$ is $*$-isomorphic to $C^*(\{u\})$, as an embedded C^*-subalgebra of $B(H_u) \overset{C^*\text{-subalgebra}}{\subseteq} B(H)$. By [13], for the given unitary part u, we can create a one-vertex-one-edge graph G. i.e., $\{u\}$ is a connected G-family in the sense of Section 2.3. Therefore,

$$C^*(\{u\}) \overset{*\text{-isomorphic}}{=} (\mathbb{C} \cdot 1_{H_u}) \otimes_{\mathbb{C}} C(\mathbb{T}).$$

(3) Assume now that the unitary part a is the zero operator. Then $a = s$, and hence the C^*-subalgebra $C^*(\{a\})$ is $*$-isomorphic to $C^*(\{s\})$, as an embedded C^*-subalgebra of $B(H_s)$. Suppose that $\varepsilon^- = k_s \in \mathbb{N}$. For example, let $k_s = 1$, and hence let $s \in PI_{H_s}(0,\, 1)$. Then this operator s is unitarily equivalent to the unilateral shift U on $l^2(\mathbb{N}_0)$ which is Hilbert-space isomorphic to H_s. It is well-known that

the unilateral shift U generates the Toeplitz algebra $\mathcal{T}(l^2(\mathbb{N}_0))$. So, the shift part s generates C^*-algebra $*$-isomorphic to the classical Toeplitz algebra $\mathcal{T}(H_s)$.

Suppose $k_s > 1$ in $\mathbb{N}$. Then the shift part s on H_s is unitarily equivalent to the operator U^{k_s}, where U is the unilateral shift on $l^2(\mathbb{N}_0)$. Note that the operator U^{k_s} satisfies that

$$U^{k_s}\left((\xi_0,\ \xi_1,\ \xi_2,\ ...)\right) = \left(\underbrace{0,\ ....,\ 0}_{k_s\text{-times}},\ \xi_0,\ \xi_1,\ ...\right),$$

for all $(\xi_0, \xi_1, \xi_2, ...) \in l^2(\mathbb{N}_0)$, with $\dim(U^{k_s})^* = k_s$. So, the C^*-algebra $C^*(\{U^{k_s}\})$ is also $*$-isomorphic to the Toeplitz algebra $\mathcal{T}(l(\mathbb{N}_0))$. Therefore, if $k_s < \infty$ in $\mathbb{N}$, then the C^*-subalgebra $C^*(\{s\})$ is $*$-isomorphic to the classical Toeplitz algebra $\mathcal{T}(H_s)$, as an embedded C^*-subalgebra of $B(H_s) \overset{C^*\text{-subalgebra}}{\subseteq} B(H)$.

(4) Suppose $s \in PI_{H_s}(0, \infty)$. Then the operator s is unitarily equivalent to the block operator matrix

$$\begin{pmatrix} 0_{H_s} & 0_{H_s} \\ 1_{H_s} & 0_{H_s} \end{pmatrix},$$

and hence the C^*-algebra $C^*(\{s\})$ is $*$-isomorphic to $(\mathbb{G} \cdot 1_{H_s}) \otimes_{\mathbb{C}} M_2(\mathbb{C})$, by [13]. Notice that, on the previous case, the Hilbert space H_s is Hilbert-space isomorphic to $H_1 \oplus H_2$, where both H_1 and H_2 are Hilbert-space isomorphic to H_s. ∎

Motivated by the statement (2) of the previous proposition, we now define a new number $\varepsilon_u \in \mathbb{N}_0^\infty$.

Definition 3.3. *Let $a \in PI_H(\varepsilon_+,\ \varepsilon^-,\ \varepsilon_-^-)$, with its Wold decomposition $a = u + s$, where u and s are the unitary part and the shift part of a, respectively, and let H_u be the unitary part of H, in terms of a. Define the number ε_u by*

$$\varepsilon_u \overset{def}{=} \dim H_u \in \mathbb{N}_0^\infty.$$

And define the subclass $PI_H(\varepsilon_0,\ \varepsilon_+,\ \varepsilon^-,\ \varepsilon_-^-)$ of $PI_H(\varepsilon_+,\ \varepsilon^-,\ \varepsilon_-^-)$ by

$$PI_H(\varepsilon_0,\ \varepsilon_+,\ \varepsilon^-,\ \varepsilon_-^-) \overset{def}{=} \{a \in PI_H(\varepsilon_+,\ \varepsilon^-,\ \varepsilon_-^-) : \dim H_{u_a} = \varepsilon_0\},$$

where H_{u_a} is the unitary part of H, in terms of a.

Since $PI(H) = \underset{(\varepsilon_+,\,\varepsilon_-)\in\mathbb{N}_0^\infty}{\cup} PI_H(\varepsilon_+,\,\varepsilon_-)$, we can get that

$$PI(H) = \underset{(\varepsilon_0,\,\varepsilon_+,\,\varepsilon^-,\,\varepsilon_-^-)\in(\mathbb{N}_0^\infty)^4}{\cup} \left(PI_H(\varepsilon_0,\ \varepsilon_+,\ \varepsilon^-,\ \varepsilon_-^-)\right).$$

Define an equivalence relation $\mathcal{R}_*$ on $PI(H)$ by

(3.1)
$$a_1\ \mathcal{R}_*\ a_2 \overset{def}{\Longleftrightarrow} C^*(\{a_1\}) \overset{*\text{-isomorphic}}{=} C^*(\{a_2\}) \text{ in } B(H).$$

By the previous proposition, if $s_1 \in PI_H(0,\,k_1)$ and $s_2 \in PI_H(0,\,k_2)$ are shifts (which are isometries on H), where $k_1,\,k_2 < \infty$ in $\mathbb{N}$, then we can get that

$$\begin{aligned} C^*(\{s_1\}) &\overset{*\text{-isomorphic}}{=} C^*(\{U^{k_1}\}) \\ &\overset{*\text{-isomorphic}}{=} \mathcal{T}(H) \\ &\overset{*\text{-isomorphic}}{=} C^*(\{U^{k_2}\}) \overset{*\text{-isomorphic}}{=} C^*(\{s_2\}), \end{aligned}$$

where $\mathcal{T}(H)$ is the classical Toeplitz algebra.

Definition 3.4. *Let $a \in PI_H(\varepsilon_0,\,\varepsilon_+,\,\varepsilon^-,\,\varepsilon_-^-)$. The quadruple $(\varepsilon_0,\,\varepsilon_+,\,\varepsilon^-,\,\varepsilon_-^-) \in (\mathbb{N}_0^\infty)^4$ is called the $*$-isomorphic index of a. The $*$-isomorphic index of a is denoted by $i_*(a)$. i.e., $i_*(a) = (\varepsilon_0,\,\varepsilon_+,\,\varepsilon^-,\,\varepsilon_-^-)$.*

Consider the set

$$(\mathbb{N}_0^\infty)^4 = \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty,$$

in detail. We will define a binary operation $(-)$ on $(\mathbb{N}_0^\infty)^4$ by

$$\begin{aligned} &(i_1,\,j_1,\,k_1,\,l_1) - (i_2,\,j_2,\,k_2,\,l_2) \\ &\qquad \overset{def}{=} (|i_1 - i_2|\,,\ \ |j_1 - j_2|\,,\ \ |k_1 - k_2|\,,\ \ |l_1 - l_2|) \end{aligned}$$

under the additional rule:

$$\infty - \infty \overset{def}{=} 0 \text{ in } \mathbb{N}_0^\infty,$$

for all $(i_t,\,j_t,\,k_t,\,l_t) \in (\mathbb{N}_0^\infty)^4$, for $t = 1,\,2$. From now, if we denote $(\mathbb{N}_0^\infty)^4$, then it means the algebraic pair $((\mathbb{N}_0^\infty)^4,\,-)$. Then, we can get the following theorem.

Theorem 3.3. *Let $a_j \in PI(H)$ be partial isometries, and suppose the spectra spec(u_j) of the unitary parts u_j of a_j are identical (if they are nonzero), for $j = 1$, 2. Then we can have that*

$$i_*(a_1) - i_*(a_2) = (0,\, k_1,\, k_2,\, 0) \tag{3.2}$$

for some $k_1, k_2 < \infty$ in $\mathbb{N}_0$, if and only if $a_1\ \mathcal{R}_*\ a_2$, equivalently, the C^*-algebras $C^*(\{a_1\})$ and $C^*(\{a_2\})$ are $*$-isomorphic, as embedded C^*-subalgebras of $B(H)$.

Proof. Suppose the $*$-isomorphic indices $i_*(a_k)$ are $(t_1^{(k)},\, t_2^{(k)},\, t_3^{(k)},\, t_4^{(k)})$ in $(\mathbb{N}_0^\infty)^4$, for $k = 1$, 2. Equivalently, assume that $a_k \in PI_H(t_1^{(k)},\, t_2^{(k)},\, t_3^{(k)},\, t_4^{(k)})$, for $k = 1$, 2. Suppose a_k have their Wold decomposition $u_k + s_k$, where u_k are the unitary parts of a_k and s_k are the shift parts of a_k, for $k = 1$, 2. Also, let H_{u_k} and H_{s_k} be the unitary parts of H and the shift parts of H, in terms of a_k, for $k = 1$, 2, respectively.

($\Rightarrow$) Suppose the $*$-isomorphic indices $i_*(a_1)$ and $i_*(a_2)$ satisfies (3.2). Then the condition (3.2) says that

$$t_1^{(1)} = \dim H_{u_1} = t_1 = \dim H_{u_2} = t_1^{(2)},$$

$$t_2^{(1)} = \dim(\ker a_1) = t_2 = \dim(\ker a_2) = t_2^{(2)},$$

$$t_4^{(1)} = \dim(\ker a_1^* \ominus \ker s_1^*) = t_4 = \dim(\ker a_2^* \ominus \ker s_2^*) = t_4^{(2)}$$

and

$$\left|t_2^{(1)} - t_2^{(2)}\right| = k_1,\ \left|t_3^{(1)} - t_3^{(2)}\right| = k_2 \text{ in } \mathbb{N}_0.$$

Assume now that $k_1 < \infty$ and $k_2 < \infty$. Then

$$a_1 \in PI_H(t_1,\, t_2^{(1)},\, t_3^{(1)},\, t_4) \text{ and } a_2 \in (t_1,\, t_2^{(1)},\, t_3^{(2)},\, t_4).$$

Since $C^*(\{a_j\}) \overset{*\text{-isomorphic}}{=} C^*(\{u_j\}) \oplus C^*(\{s_j\})$, for $j = 1$, 2, it suffices to show that $C^*(\{s_1\})$ and $C^*(\{s_2\})$ are $*$-isomorphic. By the previous proposition, $C^*(\{s_k\})$ are $*$-isomorphic either $(\mathbb{C} \cdot 1_{H_k}) \otimes_{\mathbb{C}} M_2(\mathbb{C})$ or $\mathcal{T}(H_{s_k})$, where $\mathcal{T}(H_{s_k})$ is the Toeplitz algebra on H_{s_k}, for $k = 1$, 2. By the assumption that

$$k_1 = \left|t_2^{(1)} - t_2^{(2)}\right| < \infty \text{ and } k_2 = \left|t_3^{(1)} - t_3^{(2)}\right| < 0,$$

either $[t_3^{(1)} < \infty$ and $t_3^{(2)} < \infty]$ or $[t_3^{(1)} = \infty = t_3^{(2)}]$. If $t_3^{(1)} < \infty$ and $t_3^{(2)} < \infty$, then

$$C^*(\{s_1\}) \overset{*\text{-isomorphic}}{=} \mathcal{T}(l^2(\mathbb{N}_0)) \overset{*\text{-isomorphic}}{=} C^*(\{s_2\}),$$

by the previous proposition. If $t_3^{(1)} = t_3^{(2)}$, then

$$C^*(\{s_1\}) \overset{*\text{-isomorphic}}{=} (\mathbb{C} \cdot 1_{H_0}) \otimes_{\mathbb{C}} M_2(\mathbb{C}) \overset{*\text{-isomorphic}}{=} C^*(\{s_2\}),$$

again by the previous proposition. Therefore, if (3.2) holds true, then $C^*(\{s_1\})$ and $C^*(\{s_2\})$ are $*$-isomorphic, and hence, $C^*(\{a_1\})$ and $C^*(\{a_2\})$ are $*$-isomorphic, as embedded C^*-subalgebras of $B(H)$. i.e., $a_1 \ \mathcal{R}_* \ a_2$.

($\Leftarrow$) Assume now that two partial isometries $a_1, a_2 \in PI(H)$ satisfy $a_1 \ \mathcal{R}_* \ a_2$, equivalently, the C^*-subalgebras $C^*(\{a_1\})$ and $C^*(\{a_2\})$ are $*$-isomorphic in $B(H)$. This means that $C^*(\{u_1\})$ and $C^*(\{u_2\})$ (resp., $C^*(\{s_1\})$ and $C^*(\{s_2\})$) are $*$-isomorphic, by the previous proposition. It is clear that $C^*(\{u_1\})$ and $C^*(\{u_2\})$ are $*$-isomorphic, as embedded C^*-subalgebras of $B(H)$, if and only if H_{u_1} is Hilbert-space isomorphic to H_{u_2} if and only if $t_1^{(1)} = t_1^{(2)}$, whenever $spec(u_1) = spec(u_2)$ in $\mathbb{T} \subset \mathbb{C}$. Clearly, as partial isometries, if $a_1 \ \mathcal{R}_* \ a_2$, then $t_4^{(1)} = t_4^{(2)}$.

Assume that both $C^*(\{s_1\})$ and $C^*(\{s_2\})$ are $*$-isomorphic to the classical Toeplitz algebra $\mathcal{T}(l^2(\mathbb{N}_0))$. Then $t_2^{(1)}, t_2^{(2)}, t_3^{(1)}, t_3^{(2)} < \infty$ in $\mathbb{N}_0^{\infty}$. Thus

$$\left|t_2^{(1)} - t_2^{(2)}\right| = k_1 < \infty \text{ and } \left|t_3^{(1)} - t_3^{(2)}\right| = k_2 < \infty.$$

Assume now that both $C^*(\{s_1\})$ and $C^*(\{s_2\})$ are $*$-isomorphic to the C^*-algebra $(\mathbb{C} \cdot 1_K) \otimes_{\mathbb{C}} M_2(\mathbb{C})$. Then $t_3^{(1)} = \infty = t_3^{(2)}$ in $\mathbb{N}_0^{\infty}$. Thus

$$\left|t_3^{(1)} - t_3^{(2)}\right| = 0 = \left|t_2^{(1)} - t_2^{(2)}\right|,$$

by the rule $\infty - \infty \overset{def}{=} 0$ in $\mathbb{N}_0^{\infty}$. This shows that if $a_1 \ \mathcal{R}_* \ a_2$ and if $i_*(a_k) = (t_1^{(k)}, t_2^{(k)}, t_3^{(k)}, t_4^{(k)})$ in $(\mathbb{N}_0^{\infty})^4$, then

$$i_*(a_1) - i_*(a_2) = (0, k_1, k_2, 0), \text{ with } k_1, k_2 < \infty.$$

∎

The above theorem shows that the $*$-isomorphic indexing

$$i_*(\cdot) : PI(H) \to (\mathbb{N}_0^\infty)^4$$

is *a kind of* invariant on $*$-isomorphic C^*-subalgebras generated by a single partial isometry in $B(H)$, up to the spectrums of unitary parts and the "finite" difference of the second and the third entries of $i_*(\cdot)$.

Example 3.1. *Let $H = l^2(\mathbb{N}_0)$, and let $a \in B(H)$ be an operator which is unitarily equivalent to the following matrix*

$$\begin{pmatrix} \begin{pmatrix} \theta & & \\ & \ddots & \\ & & \theta \end{pmatrix} & & 0 \\ & \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & \ddots & \ddots \end{pmatrix} & \\ 0 & & \begin{pmatrix} 0 & & \\ & 0 & \\ & & \ddots \end{pmatrix} \end{pmatrix},$$

where $\theta \in \mathbb{T}$ in $\mathbb{C}$. Without loss of generality, let a be the operator having above matrix form on H. Then it is a partial isometry on H. Assume that the left cornered block matrix

$$\begin{pmatrix} \theta & & \\ & \ddots & \\ & & \theta \end{pmatrix} \overset{denote}{=} a_u$$

of a is contained in the matricial algebra $M_n(\mathbb{C})$, with $spec(a_u) = \{\theta\}$, and the middle block matrix

$$\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \\ & & & \ddots \end{pmatrix} \overset{denote}{=} a_s$$

is the unilateral shift. Then this partial isometry a has its Wold decomposition $a = a_u + a_s$, and the unitary part H_{u_a} of H is Hilbert-space isomorphic to $\mathbb{C}^{\oplus\, n}$ and the shift part H_{a_s} of H is Hilbert-space isomorphic to $l^2(\mathbb{N}_0)$. So, we can get that this partial isometry a is contained in $PI_H(n, \infty, 1, \infty)$, and hence the $*$-isomorphic index $i_*(a)$ is

$$i_*(a) = (n,\, \infty,\, 1,\, \infty).$$

Also, the C^*-subalgebra $C^*(\{a\})$ generated by a is $*$-isomorphic to

$$\begin{aligned}\mathbb{C}^{\oplus\, n} \quad & \oplus\ \mathcal{T}\left(l^2(\mathbb{N}_0)\right) \\ & \overset{C^*\text{-subalgebra}}{\subseteq} \quad B(H_{a_u}) \oplus B(H_{a_s}) \\ & \overset{C^*\text{-subalgebra}}{\subseteq} \quad B(H).\end{aligned}$$

3.2. **Graph Groupoids Induced by Partial Isometries.**

Throughout this section, we will use the same notations we used in the previous sections. In Section 3.1, we characterized C^*-subalgebras of $B(H)$ generated by a single partial isometry. In the rest of this Section, we will observe C^*-subalgebras of $B(H)$ generated by multi-partial isometries on H. So, let $\mathcal{G} = \{a_1, ..., a_N\}$ be a family of partial isometries a_j's on H, for all $j = 1, ..., N$. And let's assume that $N > 1$. Also, let

$$i_*(a_j) = (k_1^{(j)},\, k_2^{(j)},\, k_3^{(j)},\, k_4^{(j)}), \text{ for all } j = 1, ..., N.$$

i.e.,

$$k_1^{(j)} = \dim H_{u_j}, \text{ where } u_j \text{ is the unitary part of } a_j,$$

$$k_2^{(j)} = \dim\left(\ker a_j^*\right), \quad k_3^{(j)} = \dim\left(\ker s_j^*\right)$$

and

$$k_4^{(j)} = \dim\left(\ker a_j^* \ominus \ker s_j^*\right),$$

for all $j = 1, ..., N$, where a_j have their Wold decomposition $a_j = u_j + s_j$, for $j = 1, ..., N$. Then we can construct a family $\mathcal{G}_W$ induced by $\mathcal{G}$:

$$\mathcal{G}_W = \mathcal{G}^{(u)} \cup \mathcal{G}^{(s)},$$

where

$$\mathcal{G}^{(u)} \overset{def}{=} \{u_j : j = 1, ..., N\}$$

and

$$\mathcal{G}^{(s)} \overset{def}{=} \{s_j : j = 1, ..., N\}.$$

Also, for our purpose, we decompose $\mathcal{G}^{(s)}$ by

$$\mathcal{G}^{(s)} = \mathcal{G}_f^{(s)} \cup \mathcal{G}_\infty^{(s)},$$

where

$$\mathcal{G}_f^{(s)} \overset{def}{=} \left\{ x \in \mathcal{G}^{(s)} \;\middle|\; \begin{array}{c} i_*(x) = (0, \varepsilon_+, \varepsilon^-, \varepsilon_-^-), \\ \varepsilon_+,\ \varepsilon_-^- \in \mathbb{N}_0^\infty, \text{ and} \\ \varepsilon^- < \infty \text{ in } \mathbb{N}_0 \end{array} \right\}$$

and

$$\mathcal{G}_\infty^{(s)} \overset{def}{=} \left\{ x \in \mathcal{G}^{(s)} \;\middle|\; \begin{array}{c} i_*(x) = (0,\ \varepsilon_+,\ \varepsilon^-,\ \varepsilon_-^-) \\ \varepsilon_+,\ \varepsilon_-^- \in \mathbb{N}_0^\infty, \text{ and} \\ \varepsilon^- = \infty \end{array} \right\}.$$

Define a subset $\widehat{\mathcal{G}_W}$ of $B(H)$ by $\mathcal{G}_W \cup \mathcal{G}_W^*$. Also, define the set $\mathcal{G}_{pro}$ induced by $\mathcal{G}_W$ by

$$\mathcal{G}_{pro} \overset{def}{=} \{x^* \ x,\ x \ x^* : x \in \mathcal{G}_W\} \subset B(H)_{pro},$$

where $B(H)_{pro}$ is the subset of $B(H)$ consisting of all projections on H.

Consider now the subset $\mathbb{G}_0^0 \subset B(H)$, consisting of words in $\{x^{k\,*} \ x^k : x \in \widehat{\mathcal{G}_W}, k \in \mathbb{N}\}$. i.e.,

$$\mathbb{G}_0^0 \overset{def}{=} \{0\} \cup \mathcal{G}_{pro} \cup \left(\bigcup_{k=2}^{\infty} \{(x^k)(x^k)^*,\ (x^k)^*(x^k) : x \in \mathcal{G}_W\} \right)$$

$$= \{0\} \cup \left(\bigcup_{k=1}^{\infty} \left(\{(x^k)(x^k)^*,\ (x^k)^*(x^k) : x \in \mathcal{G}_W\} \right) \right)$$

Notice that, if $x \in \widehat{\mathcal{G}^{(u)}} \subset \widehat{\mathcal{G}_W}$, where $\widehat{\mathcal{G}^{(u)}} = \mathcal{G}^{(u)} \cup \mathcal{G}^{(u)\,*}$, then

$$x^* \ x = x \ x^* = 1_{H_u} = (x^n) \ (x^n)^* = (x^n)^* \ (x^n), \text{ for all } n \in \mathbb{N}.$$

If $x \in \widehat{\mathcal{G}_\infty^{(s)}} = \mathcal{G}_\infty^{(s)} \cup \mathcal{G}_\infty^{(s)\,*}$, then

$$x^n = 0 = (x^n)^* = (x^*)^n, \text{ for all } n \in \mathbb{N} \setminus \{1\}.$$

Suppose now that $x \in \widehat{\mathcal{G}_f^{(s)}} = \mathcal{G}_f^{(s)} \cup \mathcal{G}_f^{(s)\,*}$. Then we can have that

$$(x^{n_1}) \ (x^{n_1})^* \neq (x^{n_2})(x^{n_2})^* \text{ in } B(H)_{pro}, \text{ for all } n_1 \neq n_2 \in \mathbb{N}.$$

Remark 3.1. *Suppose* $x \in \mathcal{G}_f^{(s)}$. *Then,* $x^n \in B(H)$ *is a shift, too. Indeed, if* $i_*(x) = (0,\ \varepsilon_+,\ k,\ \varepsilon_-^-)$, *with* $k < \infty$ *in* $\mathbb{N}_0$, *then* $i_*(x^n) = (0,\ \varepsilon_+,\ nk,\ \varepsilon_-^-)$ *in* $(\mathbb{N}_0^\infty)^4$. *This shows that the operators* $(x^n)\ (x^n)^*$ *are projections on* H, *too, since* x^n *are partial isometries on* H, *for all* $n \in \mathbb{N}$.

By the previous remark, we can conclude that:

Lemma 3.4. *Let* $p \in \mathbb{G}_0^0$. *Then* p *is a projection on* H.

Proof. Let $p \in \mathbb{G}_0^0$. Then there exists $x \in \widehat{\mathcal{G}_W}$ and $m \in \mathbb{N}$, such that $p = (x^m)\ (x^m)^*$. Assume that $x \in \widehat{\mathcal{G}^{(u)}}$. Then

$$(x^m)\ (x^m)^* = 1_{H_x} = x\ x^* = x^*\ x,$$

for all $m \in \mathbb{N}$, where $H_x = (x\ x^*)\ H$ is the subspace of H, which is both the initial and the final spaces of x. Suppose that $x \in \widehat{\mathcal{G}_\infty^{(s)}}$. Then

$$(x^m)(x^m)^* = \begin{cases} x\ x^* & \text{if } m = 1 \\ 0 & \text{otherwise,} \end{cases}$$

since $x^m = 0 = x^{*\,m}$, whenever $m > 1$. Therefore, it is a projection on H. Assume now that $x \in \widehat{\mathcal{G}_f^{(s)}}$. Then, by the previous remark, the operator $(x^m)\ (x^m)^*$ is a projection on H, too. This shows that $\mathbb{G}_0^0 \subset B(H)_{pro}$. ∎

Now, define the partial ordering $\leq$ on $\mathbb{G}_0^0$ by the rule:

$$p \leq q \overset{def}{\Longleftrightarrow} p\ H \overset{\text{Subspace}}{\subseteq} q\ H, \text{ for all } p,\ q \in \mathbb{G}_0^0.$$

The above partial ordering $\leq$ is well-defined, since all elements of $\mathbb{G}_0^0$ are projections on H.

Notation Without loss of generality, we will denote the partially ordered set $(\mathbb{G}_0^0,\ \leq)$ in $B(H)_{pro}$ simply by $\mathbb{G}_0^0$. □

3.2.1. *Construction of Corresponding Graphs of Partial Isometries.*

Let $x \in \mathcal{G}_W$. Then we can construct a corresponding directed graph G_x as follows:

(Case I) Suppose $u \in \mathcal{G}^{(u)}$. Then construct the one-vertex-one-loop-edge graph G_u by a directed graph with

$$V(G_u) = \{u^* \ u = u \ u^*\} \text{ and } E(G_u) = \{u\}.$$

i.e., we regard u as a loop-edge and we regard the projections $u^* \ u$ and $u \ u^*$ as the initial and the terminal vertices of u:

$$G_u = \quad \underset{u}{\overset{u^*u=uu^*}{\bullet \circlearrowleft}}$$

(Case II) Suppose $s \in \mathcal{G}_{\infty}^{(s)}$. Then construct the one-edge graph G_s by a directed graph with

$$V(G_s) = \{s^* \ s, \ s \ s^*\} \text{ and } E(G_s) = \{s\}.$$

i.e., we regard s, $s^* \ s$ and $s \ s^*$ as the non-loop edge and the corresponding initial and terminal vertices of the edge, respectively:

$$G_s = \quad \underset{s*s}{\bullet} \overset{s}{\longrightarrow} \underset{ss^*}{\bullet}$$

(Case III) Assume now that $x \in \mathcal{G}_f^{(s)}$. Then construct the directed graph G_x by the countable directed linear graph with

$$V(G_x) = \{x^* \ x\} \cup \{(x^n) \ (x^n)^* : n \in \mathbb{N}\}$$

and

$$E(G_x) = \{x^{(1)} = x\} \cup \{x^{(k)} : k \in \mathbb{N} \setminus \{1\}\}.$$

i.e., we can have the following graph G_x:

$$G_x = \quad \underset{x^*x}{\bullet} \overset{x}{\longrightarrow} \underset{xx^*}{\bullet} \overset{x^{(2)}}{\longrightarrow} \underset{x^2x^{2\,*}}{\bullet} \overset{x^{(3)}}{\longrightarrow} \underset{x^3x^{3\,*}}{\bullet} \overset{x^{(4)}}{\longrightarrow} \cdots$$

Notice that $x^{(n)}$'s satisfy that

$$x^{(1)} = x \text{ and } x^{(n+1)} = x = x \mid_{(x^n x^{n\,*})H}, \text{ for all } n \in \mathbb{N}$$

and

$$x^{(1)} \ x^{(2)} \ ... \ x^{(m)} = x^m \in B(H), \text{ for all } m \in \mathbb{N} \setminus \{1\},$$

which are shifts on H. Moreover, each vertex $(x^n)\ (x^n)^*$ are the projection onto the final space of the shift x^n, acting as the terminal vertex of the edge $x^{(n)}$, for all $n \in \mathbb{N} \setminus \{1\}$.

Definition 3.5. *Let G_x be the directed graphs induced by $x \in \mathcal{G}_W$, by the above rules. Denote the collection of countable directed graphs G_x's induced by x's in $\mathcal{G}_W$ by $\mathcal{G}_G$. The collection $\mathcal{G}_G$ is called the (collection of) corresponding graphs induced by $\mathcal{G}_W$.*

3.2.2. *The Glued Graph $G_1\ {}^{v_1}\#^{v_2}\ G_2$ of G_1 and G_2.*

Let G_k be countable directed graphs and let $v_k \in V(G_k)$ be the fixed vertices, for $k = 1, 2$. Then, by identifying the fixed two vertices v_1 and v_2, we can construct a new countable directed graph $G = G_1\ {}^{v_1}\#^{v_2}\ G_2$, by a countable directed graph

$$V(G) = (V(G_1)\ \setminus\ \{v_1\}) \cup \{v_{12}\} \cup (V(G_2)\ \setminus\ \{v_2\})$$

and

$$E(G) = E(G_1) \cup E(G_2),$$

where v_{12} is the identified vertex of the vertices v_1 and v_2 (after identifying v_1 and v_2). If $e_k \in E(G_k)$ and if $e_k = v_k\ e_k$ or $e_k = e_k\ v_k$, then this edge e_k is regarded as $e_k = v_{12}\ e_k$, respectively, $e_k = e_k\ v_{12}$ in $E(G)$. For instance, let

$$G_1 = \quad \bullet \longrightarrow \overset{v_1}{\bullet} \longrightarrow \bullet$$

and

$$G_2 = \quad \bullet \longrightarrow \overset{v_2}{\bullet}.$$

Then the graph $G = G_1\ {}^{v_1}\#^{v_2}\ G_2$ is the graph

$$G = \quad \begin{array}{ccccc} \bullet & \longrightarrow & \overset{v_{12}}{\bullet} & \longrightarrow & \bullet \\ & & \uparrow & & \\ & & \bullet & & \end{array} .$$

Definition 3.6. *The graph $G = G_1\ {}^{v_1}\#^{v_2}\ G_2$ is called the glued graph of G_1 and G_2 by gluing (or identifying) v_1 and v_2. The vertex v_{12} of v_1 and v_2 in $V(G)$ is said to be the identified (or glued) vertex.*

Inductively, we can construct the iterated glued graph G of the countable directed graphs K and T by

$$G = (K\ {}^{v_1}\#^{v_2}\ T)\ {}^{v_3}\#^{v_4}\ T\ {}^{v_5}\#^{v_6}\ T\ ...$$

etc.

As usual, we can create the shadowed graph $\widehat{G}$ and the corresponding graph groupoid $\mathbb{G}$ of the glued graph G. Notice that

$$G_1\ {}^{v_1}\#^{v_2}\ G_2 = G_2\ {}^{v_2}\#^{v_1}\ G_1.$$

3.2.3. *Conditional Iterated Gluing on $\mathcal{G}_G$.*

Let $\mathcal{G}_G$ be the corresponding graphs induced by the Wold decomposed family $\mathcal{G}_W$ of $\mathcal{G}$ in $PI(H)$. Define now the subset $\mathbb{G}_0$ of $B(H)$ by

$$\mathbb{G}_0 \overset{def}{=} \{\text{all reduced words in } \mathbb{G}_0^0\} \subset B(H).$$

i.e., if $y \in \mathbb{G}_0$ is nonzero, then there exist $n \in \mathbb{N}$ and $p_1, ..., p_n \in \mathbb{G}_0^0$ such that

$$y = \text{the operator formed by the product } p_1\ ...\ p_n.$$

Remark that, even though $p_1, ..., p_n$ are projections on H, we cannot guarantee that the operator y is a projection on H. Recall that the operator product $p\ q$ of projections p and q is a projection if and only if $p\ q = q\ p$ on H.

First, we will determine the connecting relation on $\widehat{\mathcal{G}_W} = \mathcal{G}_W \cup \mathcal{G}_W^*$ on H, by defining the map

$$\pi : \widehat{\mathcal{G}_W} \times \widehat{\mathcal{G}_W} \to \mathbb{G}_0$$

by

$$\pi(x,\ y) \overset{def}{=} (x^*\ x)\ (y\ y^*),$$

for all $x,\ y \in \widehat{\mathcal{G}_W}$. Clearly, we can get that

$$\pi(x,\ y) = \begin{cases} x^*\ x & \text{if } x^*\ x \leq y\ y^* \text{ in } \mathbb{G}_0^0 \\ y\ y^* & \text{if } x^*\ x \geq y\ y^* \text{ in } \mathbb{G}_0^0 \\ (x^*\ x)(y\ y^*) & \text{if } (x^*\ x)\ (y\ y^*) \neq 0 \text{ on } H \\ 0 & \text{otherwise,} \end{cases}$$

for all $x,\ y \in \widehat{\mathcal{G}_W}$. Consider that if $\pi(x,\ y) \neq 0$ in $\mathbb{G}_0$, then

$$x\ y = x\ (x^*\ x)\ (y\ y^*)\ y = x\ \pi(x,\ y)\ y \neq 0 \text{ on } H.$$

Definition 3.7. *The map* $\pi : \widehat{\mathcal{G}_W} \times \widehat{\mathcal{G}_W} \rightarrow \mathbb{G}_0$ *is said to be the* $\mathcal{G}_W$*-admissibility map.*

Now, define a new subset $\mathbb{G}$ of $B(H)$ by the collection of all reduced words in $\widehat{\mathcal{G}_W}$. i.e.,

$$\mathbb{G} \overset{def}{=} \{\text{all reduced words in } \widehat{\mathcal{G}_W}\} \subset B(H).$$

The reduction on $\mathbb{G}$ is determined by the operator multiplication on $B(H)$, and this reduction is totally explained by the $\mathcal{G}_W$-admissibility map π:

$$x_1, \, ..., \, x_n \in \widehat{\mathcal{G}_W} \text{ and } w = x_1 \, ... \, x_n \in \mathbb{G} \setminus \{0\}$$

$$\Longleftrightarrow \pi(x_i, \, x_{i+1}) \neq 0 \text{ in } \mathbb{G}_0, \text{ for all } i = 1, \, ..., \, n-1.$$

Notice that every element $y \in \mathbb{G}_0$ is contained in $\mathbb{G}$, too. i.e.,

$$\mathbb{G}_0 \subseteq \mathbb{G}.$$

Thus we can extend the $\mathcal{G}_W$-admissibility map π on $\widehat{\mathcal{G}_W}$ to that on $\mathbb{G}$, also denoted by π:

$$\pi : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}_0$$

defined by

$$\pi(w_1, \, w_2) \overset{def}{=} \begin{cases} \pi(w_1, \; w_2) & \text{if } (w_1, \; w_2) \in \widehat{\mathcal{G}_W} \times \widehat{\mathcal{G}_W} \\ w_1 \; w_2 & \text{if } (w_1, \; w_2) \in \mathbb{G}_0 \times \mathbb{G}_0 \\ w_1 \; (x_1 \; x_1^*) & \begin{array}{l} \text{if } w_1 \in \mathbb{G}_0 \text{ and } w_2 \notin \mathbb{G}_0 \\ \text{and } w_2 = x_1 \; ... \; x_n, \; x_j \in \widehat{\mathcal{G}_W} \end{array} \\ & \\ (x_n^* \; x_n) \; w_2 & \begin{array}{l} \text{if } w_1 \notin \mathbb{G}_0 \text{ and } w_2 \in \mathbb{G}_0 \\ \text{and } w_1 = x_1 \; ... \; x_n, \; x_j \in \widehat{\mathcal{G}_W} \end{array} \\ & \\ \pi(x_n, \; y_1) & \begin{array}{l} \text{if } w_1, \; w_2 \notin \mathbb{G}_0 \text{ in } \mathbb{G}, \text{ and} \\ w_1 = x_1 \; ... \; x_n \text{ and } w_2 = y_1 \; ... \; y_m, \\ \text{where } x_j, \; y_i \; \in \; \widehat{\mathcal{G}_W}, \end{array} \end{cases}$$

for all $w_1, \, w_2 \in \mathbb{G}$.

Denote now the collection of the shadowed graphs $\widehat{G_x}$'s of $G_x \in \mathcal{G}_G$ by $\widehat{\mathcal{G}_G}$. i.e.,

$$\widehat{\mathcal{G}_G} \overset{def}{=} \{\widehat{G_x} : G_x \in \mathcal{G}_G\}.$$

Take G_1 and G_2 in $\mathcal{G}_G$ and choose $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. For the chosen vertices v_1 and v_2, we can compute

$$\pi(v_1,\, v_2) = v_1\ v_2 \in \mathbb{G}_0 \text{ in } B(H),$$

since $V(G_x) \in \mathbb{G}_0^0 \subset \mathbb{G}_0$, for all $G_x \in \mathcal{G}_G$.

Observe now the partial ordering on $\mathcal{G}_G$. Define

$$G_1 \leq G_2 \text{ in } \mathcal{G}_G \overset{def}{\Longleftrightarrow} G_1 \text{ is a full-subgraph of } G_2.$$

Recall that we say that K_1 is a full-subgraph of K_2, where K_1 and K_2 are countable directed graphs, if

$$E(K_1) \subseteq E(K_2),$$

and

$$V(K_1) = \{v,\, v' \in V(K_2) : e = v\ e\ v',\, \forall\ e \in E(K_1)\}.$$

Remark the difference between subgraphs and full-subgraphs. We say that K_1 is a subgraph of K_2 if

$$V(K_1) \subseteq V(K_2)$$

and

$$E(K_1) = \{e \in E(K_2) : e = v\ e\ v',\, \forall\ v,\, v' \in V(K_1)\}.$$

Our partial ordering $\leq$ on $\mathcal{G}_G$ is determined by the concept, "full"-subgraphs.

Notation From now, if we denote $\mathcal{G}_G$, then it means the partial ordered set $(\mathcal{G}_G, \leq)$. $\square$

We now can construct the (π-depending) conditional glued graph $G_1\ {}^{v_1}\#_{\pi}^{v_2}\ G_2$, by a directed graph,

$$G_1\ {}^{v_1}\#_{\pi}^{v_2}\ G_2 \overset{def}{=} \begin{cases} G_1\ {}^{v_1}\#^{v_2}\ G_2 & \text{if } \pi(v_1,\ v_2) \neq 0 \\ G_2 & \text{if } G_1 \leq G_2 \\ G_1 & \text{if } G_1 \geq G_2 \\ G_1\ \cup\ G_2 & \text{otherwise,} \end{cases}$$

where $G_1 \cup G_2$ is the directed graph with

$$V\left(G_1 \cup G_2\right) = V(G_1) \sqcup V(G_2)$$

and

$$E\left(G_1 \cup G_2\right) = E(G_1) \sqcup E(G_2),$$

where the symbol "$\sqcup$" on the right-hand sides means the disjoint union.

Now, consider the conditional gluing $\#_\pi$ on $\mathcal{G}_G$ as follows: if $G_1, G_2 \in \mathcal{G}_G$ and if

$$V(G_1) = \{p_1, p_2, p_3, ...\} \text{ and } V(G_2) = \{q_1, q_2, q_3, ...\},$$

in $\mathbb{G}_0^0$, then the conditional glued graph $G_1 \#_\pi G_2$ of G_1 and G_2 is defined by

(3.3)

$$G_1 \ {}^{p_1}\#_\pi^{q_1} \ G_2 \ {}^{p_1}\#_\pi^{q_2} \ G_2 \ {}^{p_1}\#_\pi^{q_3} \ G_2 \ ...$$

$$G_1 \ {}^{p_2}\#_\pi^{q_1} \ G_2 \ {}^{p_2}\#_\pi^{q_2} \ G_2 \ {}^{p_2}\#_\pi^{q_3} \ G_2 \ ...$$

...

$$G_1 \ {}^{p_n}\#_\pi^{q_1} \ G_2 \ {}^{p_n}\#_\pi^{q_2} \ G_2 \ {}^{p_n}\#_\pi^{q_3} \ G_2 \ ...$$

....

Definition 3.8. *The graph (3.3) of G_1 and G_2 in $\mathcal{G}_G$ is denoted by $G_1 \#_\pi G_2$. And we call it the conditional (or the π-dependent) glued graph of G_1 and G_2. And the symbol "$\#_\pi$" is called the conditional (or the π-dependent) gluing.*

By the conditional *iterated* gluing on $\mathcal{G}_G$, we can decide the countable directed graph $G_{\mathcal{G}_W}$ as a directed graph with

(3.4)

$$G_{\mathcal{G}_W} \overset{def}{=} \underset{x \in \mathcal{G}_W}{\#_\pi} G_x.$$

Definition 3.9. *The conditional iterated glued graph $G_{\mathcal{G}_W}$ defined in (3.4) is called the $\mathcal{G}$-graph. And the corresponding graph groupoid $\mathbb{G}_{\mathcal{G}_W}$ of $G_{\mathcal{G}_W}$ is called the $\mathcal{G}$-groupoid.*

Notice that every element in $\mathbb{G}_{\mathcal{G}_W}$ is the reduced words in $\widehat{\mathcal{G}_W}$, under the operator multiplication on $B(H)$. Thus we can have the following lemma.

Lemma 3.5. $\mathbb{G}_{\mathcal{G}_W} = \mathbb{G}$, *where* $\mathbb{G} \overset{def}{=} \{$*all reduced words in* $\widehat{\mathcal{G}_W}\}$ *defined at the beginning of this subsection.* $\square$

Example 3.2. *Suppose that* $\mathcal{G}_W = \mathcal{G}^{(u)} \cup \mathcal{G}_\infty^{(s)}$ *and assume that*

$$\pi(x, y) \neq 0, \textit{ for } (x, y) \in \mathcal{G}_W \times \mathcal{G}_W \Longleftrightarrow x^* \, x = y \, y^* \textit{ in } \mathbb{G}_0^0.$$

Then this finite family $\mathcal{G}_W$ of partial isometries in $B(H)$ is a $G_{\mathcal{G}_W}$-family in the sense of [13] (Also See Section 2.3).

Example 3.3. *Let $\mathcal{G} = \{a\}$ and assume that $\mathcal{G}_W = \{u, s\}$, where a has its Wold decomposition $a = u + s$, with $u \neq 0$ and $s \neq 0$ on H, equivalently, the $*$-isomorphic index $i_*(a)$ of a is (k_1, k_2, k_3, k_4) in $\mathbb{N}_0^\infty$, satisfying that $k_1 \neq 0$, $k_3 = \infty$ in $\mathbb{N}_0^\infty$. Then we can create a family $\mathcal{G}_G = \{G_u, G_s\}$ of directed graphs. In particular,*

$$V(G_u) = \{u^*\ u = u\ u^*\} \text{ and } E(G_u) = \{u\}$$

and

$$V(G_s) = \{s^*\ s,\ s\ s^*\} \text{ and } E(G_s) = \{s\}.$$

Assume that

$$\pi(u,\ s) = (u^*\ u)\ (s\ s^*) = (u\ u^*)\ (s\ s^*) = \pi(u^*,\ s) \neq 0,$$

and

$$\pi(u,\ s^*) = (u^*\ u)\ (s^*\ s) = (u\ u^*)\ (s^*\ s) = \pi(u^*,\ s^*) = 0.$$

Then we can construct the conditional iterated glued graph $G = G_u \ \#_\pi\ G_s$,

$$G = G_u \ \#_\pi\ G_s = \left(G_u \ {}^{uu^*}\#^{ss^*} G_s\right),$$

Then this graph G is graph-isomorphic to the graph K,

$$K = \underset{\circlearrowleft}{\bullet} \quad \longleftarrow \quad \bullet \ .$$

Example 3.4. *Let $\mathcal{G} = \{u_1, u_2\} = \mathcal{G}^{(u)}$ be a finite family of partial isometries in $B(H)$. Then we can have $\mathcal{G}_G = \{G_{u_1}, G_{u_2}\}$, where G_k are the graph with*

$$V(G_k) = \{u_k^*\ u_k = u_k\ u_k^*\} \text{ and } E(G_k) = \{u_k\},$$

for $k = 1, 2$. Suppose $\pi(u_1, u_2) = \pi(u_1^*, u_2) = \pi(u_1, u_2^*) = \pi(u_1^*, u_2^*) \neq 0$ in $\mathbb{G}_0^0$. Then we can construct the iterated glued graph $G = G_{u_1} \ \#_\pi\ G_{u_2}$ which is identified with $G_{u_1} \ {}^{u_1^* u_1}\#_\pi^{u_2^* u_2}\ G_{u_2}$. Then this graph G is graph-isomorphic to the graph K with

$$V(K) = \{v\} \text{ and } E(K) = \{e_1 = v\ e_1\ v,\ e_2 = v\ e_2\ v\}.$$

Assume now that $\pi(u_1, u_2) = 0$ in $\mathbb{G}_0^0$. Then the iterated glued graph $G = G_{u_1} \ \#_\pi\ G_{u_2}$ is identified with the graph $G_{u_1} \sqcup G_{u_2}$. This graph is graph-isomorphic to the graph H with

$$V(H) = \{v_1,\ v_2\} \text{ and } E(H) = \{e_1 = v_1\ e_1\ v_1,\ e_2 = v_2\ e_2\ v_2\}.$$

Example 3.5. *Let $\mathcal{G} = \{s_1, s_2\} = \mathcal{G}_{\infty}^{(s)}$. Then we have the corresponding family $\mathcal{G}_G$ $= \{G_{s_1}, G_{s_2}\}$ of directed graphs. Suppose $\pi(s_1, s_2) \neq 0$, in $\mathbb{G}$. Then the conditional iterated glued graph $\Delta = G_{s_1} \#_{\pi} G_{s_2}$ is graph-isomorphic to the graph Δ:*

$$\Delta = \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ & & \downarrow \\ & & \bullet \end{array}.$$

Here, we can understand the horizontal part

$$\bullet \longrightarrow \bullet$$

of Δ are graph-isomorphic to G_{s_2}, and the vertical part

$$\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}$$

of Δ are graph-isomorphic to G_{s_1}.

The more examples would be provided in Section 5. In the following section, we will observe how the $\mathcal{G}$-groupoid $\mathbb{G}_{\mathcal{G}_W}$ induced by the $\mathcal{G}$-graph $G_{\mathcal{G}_W}$ works on the fixed Hilbert space H.

3.2.4. *The Vertex Set of the $\mathcal{G}$-Graph.*

In this subsection, we observe the vertex set $V(G)$ of the $\mathcal{G}$-graph G, which is the conditional iterated glued graph of $\mathcal{G}_G = \{G_x : x \in \mathcal{G}_W\}$. Suppose $x, y \in \mathcal{G}_W$, and assume that $\pi(x, y) \neq 0$ in $\mathbb{G}_0$. Then we can identify the vertices $x^* x \in V(G_x)$ and $y\ y^* \in G_y$, and we can create the glued graph,

$$G_{x,y} = G_x \ {}^{x^*x}\#_{\pi}^{yy^*} \ G_y,$$

which is a full-subgraph of the $\mathcal{G}$-graph G. Then, inside $G_{x,y}$, the vertices x^*x $\in V(G_x)$ and $y\ y^* \in V(G_y)$ are identified. Denote the identified vertex by p_0. Then, how we can understand this vertex p_0? Combinatorially (in the sense of Subsection 3.2.1), we can simply understand p_0 is the identified vertex which is a pure combinatorial object. However, operator-theoretically, it is not clear to see how this vertex p_0 works as an operator on H. In fact, this vertex p_0 is not determined uniquely as an operator. Operator-theoretically, we can understand p_0 as:

$$p_0 = x^*\ x \text{ or } p_0 = y\ y^* \text{ or } p_0 = \pi(x, y)$$

or

$$p_0 = \pi(x,\, y)^* = \pi(y^*,\, x^*),$$

case by case. Notice that

$$\pi(x,\, y) \neq 0 \text{ in } \mathbb{G}_0 \text{ if and only if } \pi(y^*,\, x^*) \neq 0 \text{ in } \mathbb{G}_0,$$

since $\pi(x,\, y)^* = \pi(y^*,\, x^*)$. Anyway, the vertex p_0 just represents the connection on $G_{x,\,y}$ (or on G), in terms of the $\mathcal{G}_W$-admissibility (map π).

The best way to choose $p_0 \in V(G_{x,y}) \subset V(G)$, as an operator (*if we have to: but we have not to*), is that: p_0 may be the projection from H onto the subspace $H_{init}^x \cap H_{fin}^y$. So, by gluing $x^*\ x$ and $y\ y^*$ to p_0 as the projection onto $H_{init}^x \cap H_{fin}^y$ (operator-theoretically), we may construct the conditional glued graph $G_{x,y}$. However, we will not fix p_0, as an operator. We regard p_0 as a pure combinatorial object representing the connection of the operators x and y. In other words, the identified vertex p_0 can be explained case by case, differently in Operator Theory point of view.

Observation Combinatorially, the identified vertices of the $\mathcal{G}$-graph represent the $\mathcal{G}_W$-admissibility. Operator-theoretically, these vertices are understood differently case by case. For instance, the identified vertex p_0 in the text represents $x^*\ x$ or $y\ y^*$ or $\pi(x,\, y)$ or $\pi(y^*,\, x^*)$, or more depending on the $\mathcal{G}_W$-admissibility, operator-theoretically. $\square$

3.3. **A Representation of the $\mathcal{G}$-Groupoid.**

Throughout this section, we will use the same notations we used in the previous sections. We will consider a certain representation of the $\mathcal{G}$-groupoid $\mathbb{G}$, which is the graph groupoid induced by the conditional iterated glued graph, the $\mathcal{G}$-graph G. Recall that, the operator $w \in B(H)$ is contained in $\mathbb{G}$ if and only if $w = x_1$ ... x_n, for $n \in \mathbb{N}$ and x_1, ..., $x_n \in E(\widehat{G}) = \widehat{\mathcal{G}_W}$.

Let $x \in E(\widehat{G})$. Then we can construct the subspace $H_x = (x^*\ x)\ H$ of H, where H_x is a Hilbert space where $x \in \widehat{\mathcal{G}_W}$ is acting on, as a unitary or a shift. Then we can have the class $\mathcal{H}_{\widehat{\mathcal{G}_W}}$ of subspaces of H,

$$\mathcal{H}_{\widehat{\mathcal{G}_W}} \overset{def}{=} \{H_x = (x^*\ x)\ H : x \in E(\widehat{G})\}.$$

Let H_{x_1}, $H_{x_2} \in \mathcal{H}_{\widehat{\mathcal{G}_W}}$. Then we can define subspaces of H,

$$H_{x_1} \wedge_\pi H_{x_2} \text{ and } H_{x_1} \vee_\pi H_{x_2},$$

by

$$H_{x_1} \wedge_\pi H_{x_2} \overset{def}{=} \begin{cases} H_{x_1} \cap H_{x_2} & \text{if } \pi(x_1,\ x_2) \neq 0 \\ \{0_H\} & \text{otherwise.} \end{cases}$$

and

$$H_{x_1} \vee_\pi H_{x_2} \overset{def}{=} \begin{cases} \overline{span(H_{x_1} \cup H_{x_2})}^H & \text{if } \pi(x_1,\ x_2) \neq 0 \\ H_{x_1} \ \oplus \ H_{x_2} & \text{otherwise.} \end{cases}$$

Then we can define the Hilbert space $H_{\mathcal{G}}$ by

(3.5)

$$H_{\mathcal{G}} \overset{def}{=} \underset{w \in FP_r(\widehat{G})}{\vee_\pi} H_w$$

with

$$H_w \overset{def}{=} \overset{n}{\underset{j=1}{\wedge_\pi}} H_{x_j},$$

whenever

$$w = x_1 \ ... \ x_n \in FP_r(\widehat{G}), \text{ with } x_1, \ ..., x_n \in E(\widehat{G}),$$

where

$$FP_r(\widehat{G}) \overset{def}{=} \mathbb{G} \setminus \left(V(\widehat{G}) \ \cup \ \{0\} \right).$$

Clearly, the Hilbert space $H_{\mathcal{G}}$ is a well-determined subspace of H, and every operator $w \in \mathbb{G} \subset B(H)$ acts on $H_{\mathcal{G}}$. Suppose $v \in V(\widehat{G})$. Then there always exists $e \in E(\widehat{G})$ such that $(e^* \ e) \ H \overset{\text{Subspace}}{\supseteq} v \ H$ or $(e \ e^*) \ H \overset{\text{Subspace}}{\supseteq} v \ H$. So, all operators in $\mathbb{G}$ acts on the subspace $H_{\mathcal{G}}$ of H.

We now define the groupoid action α of $\mathbb{G}$ acting on $H_{\mathcal{G}}$,

$$\alpha : \mathbb{G} \to B(H_{\mathcal{G}})$$

sending w to $\alpha_w \in B(H_{\mathcal{G}})$, by

(3.6) $$\alpha_w = w, \text{ for all } w \in \mathbb{G}.$$

Then, clearly, the action α is a groupoid action of $\mathbb{G}$ on $H_{\mathcal{G}}$. Indeed,

$$\alpha_{w_1 w_2} = w_1 \ w_2 = \alpha_{w_1} \ \alpha_{w_2}, \text{ for all } w_1, \ w_2 \in \mathbb{G}.$$

We can easily check that $\alpha_{w_1 w_2} \neq 0$ if and only if $\pi(w_1, \ w_2) \neq 0$, for all w_1, w_2 $\in \mathbb{G}$. Now, we can determine a representation of the $\mathcal{G}$-groupoid $\mathbb{G}$ by the pair ($H_{\mathcal{G}}$, α).

Definition 3.10. *The representation* $(H_{\mathcal{G}},\, \alpha)$ *of the* $\mathcal{G}$*-groupoid* $\mathbb{G}$ *is called the* $\mathcal{G}$*-representation in* $B(H)$.

In the following theorem, we observe the equivalence of $\mathcal{G}$-representations.

Theorem 3.6. *Let* $\mathcal{G}_k = \{a_1^{(k)},\, ...,\, a_{N_k}^{(k)}\} \subset PI(H)$ *be the finite families of partial isometries in* $B(H)$, *and let* G_k *be the* $\mathcal{G}_k$*-graphs having the* $\mathcal{G}_k$*-groupoids* $\mathbb{G}_k$, *respectively, for* $k = 1, 2$. *Assume that (i) the shadowed graphs* $\widehat{G_k}$ *of* G_k *are graph-isomorphic, via the graph-isomorphism* $g : \widehat{G_1} \to \widehat{G_2}$, *and (ii)* $i_*(x) - i_*(g(x)) = (0, 0, 0, 0)$ *in* $\mathbb{N}_0^\infty$, *for all* $x \in E(\widehat{G_1})$. *Then the* $\mathcal{G}_k$*-representations* $(H_{\mathcal{G}_k},\, \alpha^{(k)})$ *are equivalent, for* $k = 1, 2$.

Proof. Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are the finite families of partial isometries in $B(H)$, and let $\mathcal{G}_{k:W} = \mathcal{G}_k^{(u)} \cup \mathcal{G}_k^{(s)}$, where $\mathcal{G}_k^{(u)}$ and $\mathcal{G}_k^{(s)}$ are the collection of all unitary parts and all shift parts of $\mathcal{G}_k$, respectively, for $k = 1, 2$.

Also, by the condition (i), we can conclude that the $\mathcal{G}_k$-groupoids $\mathbb{G}_k$ are groupoid-isomorphic, for $k = 1, 2$. Indeed, we can have the groupoid-isomorphism $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$:

$$\varphi(w) \overset{def}{=} \begin{cases} 0 & \text{if } w = 0 \\ g(w) & \text{if } w \in V(\widehat{G}) \cup E(\widehat{G}) \\ g(x_1)\ ...\ g(x_n) & \text{if } w = e_1\ ...\ e_n \in FP_r(\widehat{G}),\ n > 1, \end{cases}$$

in $\mathbb{G}_2$, for all $w \in \mathbb{G}_1$ (Also, see [9] and [10]). This means that algebraically the groupoidal structures of $\mathbb{G}_1$ and $\mathbb{G}_2$ are same.

By the condition (ii), the subspaces $H_x = (x^*\ x)\ H$ and $H_{g(x)} = (g(x)^*\ g(x))\ H$ are Hilbert-space isomorphic in H, for all $x \in E(G_1)$. Therefore, the Hilbert spaces $H_{\mathcal{G}_1}$ and $H_{\mathcal{G}_2}$ are Hilbert-space isomorphic, as embedded subspaces in H. i.e., we can have the Hilbert-space isomorphism $\Phi : H_{\mathcal{G}_1} \to H_{\mathcal{G}_2}$ induced by the groupoid-isomorphism $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$:

$$\Phi\,(H_w) = H_{\varphi(w)},\ \text{in } H_{\mathcal{G}_2},\ \text{for all } H_w \subset H_{\mathcal{G}_1},\ \text{for all } w \in FP_r(\widehat{G}).$$

Therefore, the Hilbert spaces $H_{\mathcal{G}_1}$ and $H_{\mathcal{G}_2}$ are Hilbert-space isomorphic, via Φ.

Since we have the following commuting diagram,

$$\begin{array}{ccc} H_{\mathcal{G}_1} & \overset{\Phi}{\longrightarrow} & H_{\mathcal{G}_2} \\ \downarrow_{\alpha_w^{(1)}} & & \downarrow_{\alpha_{\varphi(w)}^{(2)}} \\ H_{\mathcal{G}_1} & \underset{\Phi}{\longrightarrow} & H_{\mathcal{G}_2} \end{array}$$

for all $w \in \mathbb{G}_1$, the groupoid actions $\alpha^{(k)}$ satisfy that

$$\alpha^{(2)}_{\varphi(w)} = \varphi(w) = \Phi\ w\ \Phi^{-1} = \Phi\ \alpha_w\ \Phi^{-1} = \alpha^{(1)}_{\Phi\ w\ \Phi^{-1}},$$

for all $w \in \mathbb{G}_1$. Therefore, the actions $\alpha^{(1)}$ and $\alpha^{(2)}$ are equivalent. Thus the $\mathcal{G}_k$-representations $(H_{\mathcal{G}_k},\ \alpha^{(k)})$ are equivalent, for $k = 1,\ 2$. ∎

The above theorem shows that the equivalence of the $\mathcal{G}_k$-representations $(H_{\mathcal{G}_k},$ $\alpha^{(k)})$, for $k = 1,\ 2$, is determined by the combinatorial conditions of $\mathcal{G}_{k:W}$, for $k =$ 1, 2, where $\mathcal{G}_{k:W} = \mathcal{G}_k^{(u)} \cup \mathcal{G}_k^{(s)}$.

3.4. C^*-Subalgebras Generated by Partial Isometries.

As usual, throughout this section, we will use the same notations we used in the previous sections. Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$, with its Wold decomposed family $\mathcal{G}_W$. Let $x \in \mathcal{G}_W$. We already observed the C^*-subalgebra $\mathcal{A}_x \overset{denote}{=} C^*(\{x\})$ of $B(H)$, in Section 3.1. Notice that, by the very definition, $\mathcal{A}_x$ $= \mathcal{A}_{x^*}$, for all $x \in \mathcal{G}_W$. Here, the symbol "=" means "being identically same in $B(H)$".

Proposition 3.7. *(See Section 3.1) Let $x \in \widehat{\mathcal{G}_W}$ and $\mathcal{A}_x = C^*(\{x\})$. Then*

(3.7)

$$\mathcal{A}_x \overset{*\text{-isomorphic}}{=} \begin{cases} (\mathbb{C} \cdot 1_{H_x}) \otimes_{\mathbb{C}}\ C\,(spec(x)) & \text{if } i^*(x) = (k_1,\ k_2,\ 0,\ k_4), \\ & \quad \text{for } k_1,\ k_2,\ k_4 \in \mathbb{N}_0^{\infty} \\ \mathcal{T}(H_x) & \text{if } i^*(x) = (0,\ k_2,\ k_3,\ k_4) \\ & \text{for } k_2,\ k_4 \in \mathbb{N}_0^{\infty} \text{ and } k_3 \in \mathbb{N} \\ (\mathbb{C} \cdot 1_{H_x}) \otimes_{\mathbb{C}}\ M_2(\mathbb{C}) & \text{if } i_*(x) = (0,\ k_2,\ \infty,\ k_4) \\ & \quad \text{for } k_2,\ k_4 \in \mathbb{N}_0^{\infty}, \end{cases}$$

where $i_*(\cdot)$ means the $*$-isomorphic index. □

The following theorem is the main result of this paper. This shows that a C^*-subalgebra of $B(H)$ generated by finitely many partial isometries is $*$-isomorphic to a groupoid C^*-algebra.

Theorem 3.8. *Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$ and assume that it has its $\mathcal{G}$-groupoid $\mathbb{G}$. Then the C^*-algebra $C^*(\mathcal{G})$ generated by $\mathcal{G}$ is*

$*$-isomorphic to the groupoid C^*-algebra $C^*_\alpha(\mathbb{G})$, as embedded C^*-subalgebras of $B(H_\mathcal{G}) \subseteq B(H)$, where $(H_\mathcal{G},\, \alpha)$ is the $\mathcal{G}$-representation of $\mathbb{G}$.

Proof. Observe that

$$\begin{aligned} C^*(\mathcal{G}) &= C^*(\mathcal{G}_W) = C^*\left(\widehat{\mathcal{G}_W}\right) \overset{*\text{-isomorphic}}{=} \overline{\mathbb{C}[\alpha(\mathbb{G})]} \\ &= C^*_\alpha(\mathbb{G}) \overset{C^*\text{-subalgebra}}{\subseteq} B(H_\mathcal{G}) \overset{C^*\text{-subalgebra}}{\subseteq} B(H). \end{aligned}$$

The first identity of the above formulae holds true because of the Wold decomposition. And the $*$-isomorphic relation and the first $*$-subalgebra inclusion hold true, by the definition of the $\mathcal{G}$-representation $(H_\mathcal{G},\, \alpha)$ of $\mathbb{G}$. ∎

The above theorem shows that C^*-subalgebras of $B(H)$ generated by finitely many partial isometries on H are $*$-isomorphic to certain groupoid C^*-algebras.

Corollary 3.9. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the collection of finitely many partial isometries in $B(H)$ having their $\mathcal{G}$-graphs G_1 and G_2. Assume that (i) the shadowed graphs $\widehat{G_1}$ and $\widehat{G_2}$ are graph-isomorphic, via a graph-isomorphism $g : \widehat{G_1} \to \widehat{G_2}$, (ii) $i_*(x) - i_*(g(x)) = (0,\, 0,\, 0,\, 0)$ in $\mathbb{N}_0^\infty$, for all $x \in E(\widehat{G_1})$, and (iii) For any unitary parts $u \in E(G_1)$, $spec(u) = spec\,(g(u))$ in $\mathbb{T}$. Then the C^*-algebras $C^*(\mathcal{G}_1)$ and $C^*(\mathcal{G}_2)$ are $*$-isomorphic, as embedded C^*-subalgebras of $B(H)$.*

Proof. Observe that

$$C^*(\mathcal{G}_1) \overset{*\text{-isomorphic}}{=} C^*_{\alpha^{(1)}}(\mathbb{G}_1)$$

in $B(H_{\mathcal{G}_1})$, by the previous theorem, where $\mathbb{G}_k$ are the $\mathcal{G}_k$-groupoids and where $(H_{\mathcal{G}_k},\, \alpha^{(k)})$ are the $\mathcal{G}_k$-representations of $\mathbb{G}_k$, for $k = 1,\, 2$

$$\overset{*\text{-isomorphic}}{=} C^*_{\alpha^{(2)}}(\mathbb{G}_2) \overset{*\text{-isomorphic}}{=} C^*(\mathcal{G}_2),$$

of $B(H_{\mathcal{G}_2})$, by (ii) and (iii), since $(H_{\mathcal{G}_k},\, \alpha^{(k)})$'s are equivalent. ∎

4. Block Structures of $C^*(\mathcal{G})$

The main result in this section is a two-part structure theorem (Theorem 4.3 and 4.13) for the C^*-algebras introduced above. In proving them, we will need some facts and lemmas from topological (reduced) free product algebras, introduced in Section 4.1.

Throughout this Section, we will keep using the same notations we used before. Let $\mathcal{G}$ be the given family of finitely many partial isometries in $B(H)$ and let $\mathcal{G}_W$ be the corresponding Wold decomposed family of $\mathcal{G}$. We showed that the C^*-algebra $C^*(\mathcal{G})$ is $*$-isomorphic to the groupoid C^*-algebra $C^*_\alpha(\mathbb{G})$, where $\mathbb{G}$ is the $\mathcal{G}$-groupoid induced by the $\mathcal{G}$-graph G in the operator algebra $B(H_\mathcal{G}) \overset{C^*\text{-subalgebra}}{\subseteq} B(H)$, where $(H_\mathcal{G},\, \alpha)$ is the $\mathcal{G}$-representation.

4.1. **Topological Reduced Free Product Algebras.**

Let X_1 and X_2 be arbitrary sets. Then we can define the *algebraic free product* $X_1 * X_2$ of X_1 and X_2 by the set of all words in $X_1 \cup X_2$. i.e.,

$$X_1 * X_2 \overset{def}{=} \text{all words in } X_1 \cup X_2. \tag{4.1}$$

i.e., $x \in X_1 * X_2$ if and only if there exist $n \in \mathbb{N}$ and $(i_1, ..., i_n) \in \{1, 2\}^n$ and $x_{i_j} \in X_{i_j}$, for $j = 1, ..., n$, such that

$$i_1 \neq i_2,\, i_2 \neq i_3,\, ...,\, i_{n-1} \neq i_n$$

and

$$x = x_{i_1}\ x_{i_2}\ ...\ x_{i_n}.$$

Recall now that we say a topological space A is a topological algebra over a scalar field $\mathbb{F}$, if it is an algebra over $\mathbb{F}$ and all operations on A are continuous under the topology for A. i.e., A is a topological algebra over a scalar field $\mathbb{F}$, if a topological space A, containing $\mathbb{F}$, satisfies

(i) A is a vector space over $\mathbb{F}$ with its vector addition $(+) : (a_1, a_2) \mapsto a_1 + a_2$ and the scalar multiplication $(\times) : (t, a_1) \mapsto ta_1$, for all $t \in \mathbb{F}$ and $a_1, a_2 \in A$,

(ii) A has a vector multiplication $(\cdot) : (a_1, a_2) \mapsto a_1\ a_2$, for all $a_1, a_2 \in A$, and it is associative,

(iii) the vector addition $(+)$ and the vector multiplication $(\cdot)$ on A are left and right distributive, and

(iv) the operations $(+)$, $(\cdot)$ and $(\times)$ are continuous, under the topology for A.

Recall also that we say A is an (algebraic) algebra over $\mathbb{F}$, if A satisfies (i), (ii) and (iii). For example, C^*-algebras and von Neumann algebras are topological algebras.

Definition 4.1. *Let A_1 and A_2 be algebraic algebras over the same scalar field $\mathbb{F}$. Then we can define their algebraic free product algebra $A_1 *_{a\lg} A_2$ of A_1 and A_2 by the algebra generated by all words in $A_1 \cup A_2$, in the algebra $A\lg_{\mathbb{F}}(A_1, A_2) = \mathbb{F}[A_1 \cup A_2]$, generated by A_1 and A_2. Notice that $A_1 *_{a\lg} A_2$ is not a topological space (even though A_1 and A_2 are topological spaces).*

By the definition of algebraic free product and algebraic free product algebras, we can get the following proposition.

Proposition 4.1. *Let A_1 and A_2 be algebraic algebras over a scalar field $\mathbb{F}$, and let $A_1 *_{a\lg} A_2$ be the algebraic free product algebra of A_1 and A_2. Assume now that $A_k = \mathbb{F}[X_k]$, where X_k are the generator set of A_k, for $k = 1, 2$. Then*

$$A_1 *_{a\lg} A_2 \overset{Algebra}{=} \mathbb{F}[X_1 * X_2],$$

where "$\overset{Algebra}{=}$" means "being algebra-isomorphic".

Proof. Suppose $A_k = \mathbb{F}[X_k]$ are the algebraic algebras generated by X_k, for $k = 1$, 2. Then we have that

$$A_1 *_{a\lg} A_2 = \mathbb{F}[A_1 * A_2]$$

by the definition of "$*_{a\lg}$"

$$= \mathbb{F}\left[\mathbb{F}[X_1] \ * \ \mathbb{F}[X_2]\right] \overset{\text{Algebra}}{=} \mathbb{F}\left[\mathbb{F}[X_1 \ * \ X_2]\right]$$

by the definition of "$*$"

$$= \mathbb{F}[X_1 * X_2].$$

∎

i.e., the above proposition shows that

$$\mathbb{F}[X_1 * X_2] \overset{\text{Algebra}}{=} \mathbb{F}[X_1] *_{A\lg} \mathbb{F}[X_2], \tag{4.2}$$

where X_1 and X_2 are arbitrary sets.

Similar to the previous construction, we can determine algebraic *reduced* free structures. Let $(X, \cdot)$ be an arbitrary algebraic pair. i.e., $(\cdot) : X \times X \to X$ is

a binary operation on X. (Notice that $(X, \cdot)$ is not necessarily be a well-known algebraic structures, for instance, a semigroup or a group, or a groupoid, etc.) Let $\mathcal{X}$ be the collection of all reduced words in X, where the reduction is totally depending on the binary operation $(\cdot)$ on X.

Definition 4.2. *Let $(X, \cdot)$ and $\mathcal{X}$ be given as in the previous paragraph. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be subsets of $\mathcal{X}$. Define the algebraic reduced free product set $\mathcal{X}_1 *^r \mathcal{X}_2$ of $\mathcal{X}_1$ and $\mathcal{X}_2$, by the subset of $\mathcal{X}$ consisting of all reduced words in $\mathcal{X}_1 \cup \mathcal{X}_2$.*

Similarly, we can determine the algebraic reduced free product algebra of algebraic algebras A_1 and A_2:

Definition 4.3. *Let A be an algebraic algebra over its scalar field $\mathbb{F}$, and let A_1 and A_2 be the algebraic subalgebras of A. Define the algebraic reduced free product algebra $A_1 *^r_{a\lg} A_2$ of A_1 and A_2 in A, by the algebra generated by all reduced words in $A_1 \cup A_2$. i.e.,*

$$A_1 *^r_{a\lg} A_2 \overset{def}{=} \mathbb{F}\left[\{\textit{all reduced words in } A_1 \ \cup \ A_2\}\right].$$

Remark that the reduction is dependent upon the vector multiplication on A.

Then, similar to the previous proposition or (4.2), we can get the following proposition.

Proposition 4.2. *Let A be an algebraic algebra over a scalar field $\mathbb{F}$ and assume that $A = \mathbb{F}[X]$. i.e., A is generated by a set X. Let A_1 and A_2 be algebraic subalgebras of A and suppose $A_k = \mathbb{F}[X_k]$, where X_k's are the subset of X. Then the algebraic reduced free product algebra $A_1 *^r_{a\lg} A_2$ in A is $*$-isomorphic to the algebraic algebra $\mathbb{F}[X_1 *^r X_2]$, where $X_1 *^r X_2$ is the reduced free product of X_1 and X_2, under the vector multiplication on A.*

Proof. First, notice that we can construct an algebraic pair $(X, \cdot)$, where $(\cdot)$ is the restricted vector multiplication on A, and we can understand X as $(X, \cdot)$. (So, the algebraic reduced free product set $X_1 *^r X_2$ is well-defined.) Similar to the proof of the previous proposition, we can get that

$$A_1 *^r_{a\lg} A_2 = \mathbb{F}[A_1 *^r A_2]$$

by the definition of "$*^r_{a\lg}$"

$$= \mathbb{F}\left[\mathbb{F}[X_1] \ *^r \ \mathbb{F}[X_2]\right] \overset{\text{Algebra}}{=} \mathbb{F}\left[\mathbb{F}[X_1 *^r X_2]\right]$$

by the definition of "$*^r$" and by the vector addition and multiplication on A

$$= \mathbb{F}[X_1 *^r X_2].$$

∎

So, the above proposition shows the relation between "$*^r$" and "$*^r_{a\lg}$":

$$\mathbb{F}\left[X_1 *^r X_2\right] \overset{\text{Algebra}}{=} \mathbb{F}[X_1] *^r_{a\lg} \mathbb{F}[X_2], \text{ inside } \mathbb{F}[X], \tag{4.3}$$

whenever X_1 and X_2 are subsets of X.

Now, we define the topological reduced free product algebras.

Definition 4.4. *Let A be a topological algebra over its scalar field $\mathbb{F}$, and let A_1 and A_2 be topological subalgebras of A. Define the topological reduced free product algebra $A_1 *^r_{top} A_2$ of A_1 and A_2 in A by the τ-closure of $A_1 *^r_{a\lg} A_2$, where τ is the topology for A. i.e.,*

$$A_1 *^r_{top} A_2 \overset{def}{=} \overline{A_1 *^r_{a\lg} A_2}^{\tau} \text{ inside } A. \tag{4.4}$$

By definition, we can get the following theorem.

Theorem 4.3. *Let A be a topological algebra over its scalar field $\mathbb{F}$, and let A_1 and A_2 be topological subalgebras of A. Assume that $A = \overline{\mathbb{F}[X]}^{\tau}$ and $A_k = \overline{\mathbb{F}[X_k]}^{\tau}$, where X is the generator set of A and the X_k's are subsets of X, for $k = 1, 2$. Then*

$$A_1 *^r_{top} A_2 \overset{Top\text{-}Algebra}{=} \overline{\mathbb{F}[X_1 *^r X_2]}^{\tau},$$

where "$\overset{Top\text{-}Algebra}{=}$" means "being topological algebra isomorphic".

Proof. Observe that

$$\begin{aligned} A_1 *^r_{top} A_2 &= \overline{\mathbb{F}[A_1 \ *^r_{a\lg} \ A_2]}^{\tau} = \overline{\mathbb{F}\left[\mathbb{F}[X_1] \ *^r_{a\lg} \ \mathbb{F}[X_2]\right]}^{\tau} \\ &= \overline{\mathbb{F}\left[\mathbb{F}[X_1 \ *^r \ X_2]\right]}^{\tau} = \overline{\mathbb{F}\left[X_1 \ *^r \ X_2\right]}^{\tau}, \end{aligned}$$

by (4.3). ∎

i.e., by the previous theorem, we have

$$\overline{\mathbb{F}[X_1]}^{\tau} *^r_{top} \overline{\mathbb{F}[X_2]}^{\tau} = \overline{\mathbb{F}\left[X_1 \ *^r \ X_2\right]}^{\tau}, \text{ in } \overline{\mathbb{F}[X]}, \tag{4.5}$$

whenever X_1, X_2 are subsets of X.

The following corollary is the direct consequence of the previous theorem.

Corollary 4.4. *Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$ and let $\mathbb{G}$ be the $\mathcal{G}$-groupoid. Let $\{\mathbb{G}_x : x \in \mathcal{G}_W\}$ be the subgroupoids of the $\mathcal{G}$-groupoid $\mathbb{G}$, where $\mathbb{G}_x$ are the graph groupoids induced by the corresponding graphs G_x of x in $\mathcal{G}_G$, for all $x \in \mathcal{G}_W$. Then*

$$C^*(\mathcal{G}) \overset{*\text{-isomorphic}}{=} \underset{x \in \mathcal{G}_W}{*^r_{top}} \left(C^*_\alpha(\mathbb{G}_x)\right) \text{ in } B(H_{\mathcal{G}}), \tag{4.6}$$

as embedded C^*-subalgebras of $B(H)$, where $(H_{\mathcal{G}}, \alpha)$ is the $\mathcal{G}$-representation of $\mathbb{G}$.

Proof. Denote $C^*_\alpha(\mathbb{G})$ and $C^*_\alpha(\mathbb{G}_x)$'s by $\mathcal{A}$ and $\mathcal{A}_x$'s, respectively.. By Section 4, we know that the C^*-subalgebra $C^*(\mathcal{G})$ of $B(H)$ is $*$-isomorphic to the groupoid C^*-algebra $\mathcal{A}$ in $B(H_{\mathcal{G}}) \subseteq B(H)$, where $(H_{\mathcal{G}}, \alpha)$ is the $\mathcal{G}$-representation of $\mathbb{G}$. It is easily checked that

$$\mathbb{G} = \underset{x \in \mathcal{G}_W}{*^r} \mathbb{G}_x,$$

Thus, we can have that

$$\begin{aligned} C^*(\mathcal{G}) &\overset{*\text{-isomorphic}}{=} \mathcal{A} = \overline{\mathbb{C}\left[\alpha(\mathbb{G})\right]} = \overline{\mathbb{C}[\mathbb{G}]} \\ &= \overline{\mathbb{C}\left[\underset{x \in \mathcal{G}}{*^r} \mathbb{G}_x\right]} \overset{*\text{-isomorphic}}{=} \underset{x \in \mathcal{G}_W}{*^r_{top}} \left(\overline{\mathbb{C}[\mathbb{G}_x]}\right) \\ \text{by (4.5)} \quad &= \underset{x \in \mathcal{G}_W}{*^r_{top}} \mathcal{A}_x. \end{aligned}$$

∎

The above corollary shows that the C^*-algebra $C^*(\mathcal{G})$ generated by a finite family $\mathcal{G}$ of partial isometries in $B(H)$ has its block structure determined by the topological reduced free product, and the blocks are $\mathcal{A}_x = C^*_\alpha(\mathbb{G}_x)$'s, for all $x \in \mathcal{G}_W$. Since $\mathcal{A}_x$'s are characterized in Section 3, we have the topological reduced free block structures and characterized blocks.

Remark 4.1. *Our topological reduced free (product) structure is basically different from the (amalgamated reduced) free structures observed in [9], [10], [11] and [13]. (Amalgamated) Reduced freeness on those papers are determined by the free probabilistic settings of Voiculescu (See [16]). However, there are some connections between them (See [13]).*

In the following section, we will consider this topological reduced free block structures of $C^*(\mathcal{G})$ more in detail.

4.2. **Topological Free Block Structures on $C^*(\mathcal{G})$.**

Let $\mathcal{G}$ be the given finite family of partial isometries in $B(H)$, and let $\mathcal{G}_G = \{G_x : x \in \mathcal{G}_W\}$ be the family of corresponding graphs induced by $\mathcal{G}_W$. Also, let G be the $\mathcal{G}$-graph induced by $\mathcal{G}_W$, which is the conditional iterated glued graph of $\mathcal{G}_G$, and let $\mathbb{G}$ be the $\mathcal{G}$-grouopoid, the graph groupoid of G. In the previous section, we showed that $\mathbb{G}$ is the topological reduced free product of $\mathbb{G}_x$'s, where $\mathbb{G}_x$ are the graph groupoids of G_x, which are the subgroupoid of $\mathbb{G}$, for all $x \in \mathcal{G}_W$, and

$$C^*(\mathcal{G}) \overset{*\text{-isomorphic}}{=} C^*_\alpha(\mathbb{G}) \overset{*\text{-isomorphic}}{=} \underset{x \in \mathcal{G}_W}{*^r_{top}} C^*_\alpha(\mathbb{G}_x).$$

This means that the C^*-subalgebra $C^*(\mathcal{G})$ of $B(H)$ has rough but characterized block structures $\{\mathcal{A}_x : x \in \mathcal{G}_W\}$, by Section 3.1. In this section, we will consider this block structure more in detail, inside $C^*(\mathcal{G})$.

Suppose $u \in \mathcal{G}^{(u)}$ and $G_u \in \mathcal{G}_G$. Then, by Subsection 3.2.1, the graph G_u is the one-vertex-one-loop-edge graph. Therefore, the graph groupoid $\mathbb{G}_u$ of G_u, which is a subgroupoid of $\mathcal{G}$-groupoid $\mathbb{G}$, is a group. Also, if $s \in \mathcal{G}^{(s)}_\infty$, then the corresponding graph $G_s \in \mathcal{G}_G$ is a one-edge graph having distinct initial and terminal vertices. So, the graph groupoid $\mathbb{G}_s$ of G_s is a finite (sub)groupoid consisting of the elements 0, s, s^*, $s^*\ s$ and $s\ s^*$. Therefore, if $x \in \mathcal{G}^{(u)} \cup \mathcal{G}^{(s)}_\infty$, then we can handle the corresponding subgroupoid induced by x relatively easy.

Lemma 4.5. *Let u_1, u_2, $u \in \mathcal{G}^{(u)}$ and s_1, s_2, $s \in \mathcal{G}^{(s)}_\infty$.*

(1) If $\pi(u_1, u_2) \neq 0$ in $\mathbb{G}_0$, then

$$C^*(\{u_1, u_2\}) \overset{*\text{-}isomorphic}{=} \overline{C^*(\{u_1\}) *_{a\lg} C^*(\{u_2\})}.$$

Thus, $C^*(\{u_1, u_2\})$ is $*$-isomorphic to

$$\overline{(\mathbb{C} \cdot 1_{H_{\{u_1,\, u_2\}}}) \otimes_{\mathbb{C}} (C\,(spec(u_1)) *_{a\lg} C\,(spec(u_2)))},$$

where $H_{\{u_1,\, u_2\}}$ is the $\{u_1, u_2\}$-Hilbert space in the sense of Section 3.

(2) if $\pi(s_1, s_2) \neq 0$ in $\mathbb{G}_0$, then

$$C^*(\{s_1,\ s_2\}) \overset{*\text{-}isomorphic}{=} \overline{C^*(\{s_1\}) *_{a\lg} C^*(\{s_2\})}.$$

Thus, $C^*(\{s_1,\ s_2\})$ is $*$-isomorphic to

$$(\mathbb{C} \cdot 1_{H_{\{s_1,\ s_2\}}}) \otimes_{\mathbb{C}} (M_2(\mathbb{C}) \ \otimes_{\mathbb{C}}\ M_2(\mathbb{C})) = (\mathbb{C} \cdot 1_{H_{\{s_1,\ s_2\}}}) \otimes_{\mathbb{C}} M_4(\mathbb{C}),$$

where $H_{\{s_1,\ s_2\}}$ is the $\{s_1,\ s_2\}$-Hilbert space in the sense of Section 3.

(3) if $\pi(u,\ s) \neq 0$ in $\mathbb{G}_0$, then

$$C^*(\{u,\ s\}) \overset{*\text{-}isomorphic}{=} \overline{C^*(\{u\}) *_{a\lg} C^*(\{s\})}.$$

So, $C^*(\{u,\ s\})$ is $*$-isomorphic to

$$\overline{(\mathbb{C} \cdot 1_{H_{\{u,\ s\}}}) \otimes_{\mathbb{C}} (C\,(spec(u)) *_{a\lg} M_2(\mathbb{C}))}.$$

Proof. By Section 4.1, we have that if $\mathcal{G}_W = \{x,\ y\}$, then

$$\begin{aligned} C^*\left(\mathcal{G}_W\right) &\overset{*\text{-isomorphic}}{=} C^*_{\alpha}(\mathbb{G}) = C^*_{\alpha}\left(\mathbb{G}_x *^r \mathbb{G}_y\right) \\ &\overset{*\text{-isomorphic}}{=} C^*_{\alpha}(\mathbb{G}_x) *^r_{top} C^*_{\alpha}(\mathbb{G}_y) \\ &\overset{def}{=} \overline{C^*_{\alpha}(\mathbb{G}_x) *^r_{a\lg}\ C^*_{\alpha}(\mathbb{G}_y)} \\ &\overset{*\text{-isomorphic}}{=} \overline{C^*(\{x\}) *^r_{a\lg} C^*(\{y\})}, \end{aligned}$$

where "$*^r_{a\lg}$" means the "algebraic **reduced** algebra free product".

(1) Suppose u_1 and u_2 are unitary parts of certain partial isometries in $B(H)$, and let G_{u_1} and G_{u_2} be the corresponding one-vertex-one-loop-edge graphs in $\mathcal{G}_G$. Then we can construct the conditional glued graph $G = G_{u_1} \#_{\pi} G_{u_2}$ of G_{u_1} and G_{u_2}, which is the $\mathcal{G}$-graph. It is graph-isomorphic to the one-vertex-two-loop-edge graph. Then $\mathcal{G}$-groupoid $\mathbb{G}$ satisfies that $\mathbb{G} = \mathbb{G}_{u_1} *^r \mathbb{G}_{u_2}$, where $\mathbb{G}_{u_k}$ are the graph groupoids of G_{u_k}, for $k = 1,\ 2$. However, by the conditional gluing on the $\mathcal{G}$-graph G, we can realize that $\mathbb{G}_{u_1} *^r \mathbb{G}_{u_2}$ is identified with $\mathbb{G}_{u_1} * \mathbb{G}_{u_2}$, where "$*$" means the "algebraic (**non-reduced**) free product". Therefore,

$$\begin{aligned} C^*\left(\{u_1,\ u_2\}\right) &\overset{*\text{-isomorphic}}{=} \overline{C^*(\{u_1\}) *^r_{a\lg}\ C^*(\{u_2\})} \\ &\overset{*\text{-isomorphic}}{=} \overline{C^*(\{u_1\})\ *_{a\lg}\ C^*(\{u_2\})}, \end{aligned}$$

where "$*_{a\lg}$" means the "algebraic *non-reduced* algebra free product".

(2) Assume now that s_1 and s_2 are shift parts in $\mathcal{G}_{\infty}^{(s)}$ of certain partial isometries in $B(H)$, and let G_{s_k} be the corresponding graphs with their graph groupoids $\mathbb{G}_{s_k}$, for $k = 1, 2$. Suppose $\pi(s_1, s_2) \neq 0$. Then we can create the conditional glued graph $G = G_{s_1} \#_{\pi} G_{s_2}$ which is graph-isomorphic to

$$\bullet \longrightarrow \bullet \longrightarrow \bullet.$$

Then we can easily check that the graph groupoid $\mathbb{G}$ of G is $\{0, s_1, s_2, s_1^*, s_2^*, s_1 s_2, s_2^* s_1^*\}$, set-theoretically. So,

$$\mathbb{G} = \mathbb{G}_{s_1} *^r \mathbb{G}_{s_2} = \mathbb{G}_{s_1} * \mathbb{G}_{s_2}.$$

Therefore, we have that

$$\begin{aligned} C^*(\{s_1, s_2\}) &\overset{*\text{-isomorphic}}{=} \overline{C^*(\{s_1\}) *_{a\lg}^r C^*(\{s_2\})} \\ &\overset{*\text{-isomorphic}}{=} \overline{C^*(\{s_1\}) *_{a\lg} C^*(\{s_2\})}. \end{aligned}$$

(3) Let $u \in \mathcal{G}^{(u)}$ and $s \in \mathcal{G}_{\infty}^{(s)}$ with their corresponding graphs $G_u, G_s \in \mathcal{G}_G$, respectively. Then similar to the previous two cases, we can get the desired result. ∎

The above lemma shows that it is relatively easy to deal with the groupoid C^*-algebraic structures in terms of their topological reduced free blocks of them, since the topological reduced free product of blocks is determined by the algebraic "non-reduced" free product of generator sets of blocks. This means that we do not need to consider the reduction on blocks induced by the elements in $\mathcal{G}^{(u)} \cup \mathcal{G}_{\infty}^{(s)}$, inside $C^*(\mathcal{G})$.

However, if x is contained in $\mathcal{G}_f^{(s)}$, then the corresponding graph G_x is graph-isomorphic to the infinite linear graph,

$$\underset{x^*x}{\bullet} \overset{x}{\longrightarrow} \underset{xx^*}{\bullet} \overset{x^{(2)}}{\longrightarrow} \underset{x^2x^{2\,*}}{\bullet} \overset{x^{(3)}}{\longrightarrow} \underset{x^3x^{3\,*}}{\bullet} \overset{x^{(4)}}{\longrightarrow} \cdots,$$

and hence it is hard to handle the graph groupoid $\mathbb{G}_x$ of G_x, where $x^{(k)} = x$ on H, for all $k \in \mathbb{N}$, with $x^{(1)} = x$, satisfying

$$x\, x^{(2)} \ldots x^{(m)} = x^m \text{ on } H, \text{ for all } m \in \mathbb{N}.$$

We will concentrate on the case where we have a fixed shift part x in $\mathcal{G}_f^{(s)}$. Let $x \in \mathcal{G}_f^{(s)}$ and $G_x \in \mathcal{G}_G$, and let $y \in \mathcal{G}_W$ with its corresponding graph $G_y \in \mathcal{G}_G$. First, assume that $y \in \mathcal{G}^{(u)}$. If $\pi(x,\, y) = 0$ in $\mathbb{G}_0$, then we can have the disconnected graph $G_{x,y} = G_x \#_\pi G_y = G_x \cup G_y$,

$$\begin{array}{l} \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \\ \\ \underset{\circlearrowleft}{\bullet} \end{array}$$

If $\pi(x,\, y) \neq 0$ in $\mathbb{G}_0$, then $\pi(x^n,\, y) \neq 0$, for all $n \in \mathbb{N}$, since

$$x^*\ x \geq x\ x^* \geq x^2\ x^{2\,*} \geq x^3\ x^{3\,*} \geq x^4\ x^{4\,*} \geq ....$$

Thus, the conditional iterated glued graph $G_{x,y}$ is

$$\begin{array}{ccccccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \cdots \end{array}.$$

If we consider the graph groupoid $\mathbb{G}_{x,y}$ of the conditional glued graph $G_{x,y}$ of G_x and G_y, then we can conclude that

$$\mathbb{G}_{x,y} = \mathbb{G}_x *^r \mathbb{G}_y = \mathbb{G}_x * \mathbb{G}_y, \tag{4.7}$$

where "$*^r$" means the reduced algebraic free product and "$*$" means the non-reduced algebraic free product.

Lemma 4.6. *Let $x \in \mathcal{G}_f^{(s)}$ and $u \in \mathcal{G}^{(u)}$, assume that $\pi(x,\, u) \neq 0$ in $\mathbb{G}_0$. Then the C^*-subalgebra $C^*(\{x,\, u\})$ of $B(H)$ is $*$-isomorphic to $\overline{C^*(\{x\}) *_{a\lg} C^*(\{y\})}$. Thus, $C^*(\{x,\, u\})$ is $*$-isomorphic to*

$$\overline{(\mathbb{C} \cdot 1_{H_{\{x,\, u\}}}) \otimes_{\mathbb{C}} (\mathcal{T}(l^2(\mathbb{N}_0))\ \ *_{a\lg}\ \ C(spec(u)))}.$$

Proof. It suffices to show that the graph groupoid $\mathbb{G}$ of the conditional glued graph $G_{x,\,y}$ of G_x and G_y satisfies $\mathbb{G} = \mathbb{G}_x * \mathbb{G}_y$, where $\mathbb{G}_x$ and $\mathbb{G}_y$ are the graph groupoids of the corresponding graphs G_x and G_y of x and y. By (4.7), it is done. ∎

Assume now that $x \in \mathcal{G}_f^{(s)}$ and $s \in \mathcal{G}_\infty^{(s)}$, and let $\pi(x,\, s) \neq 0$ in $\mathbb{G}_0$. Then, $\pi(x^n,\, s) \neq 0$ in $\mathbb{G}_0$, too, for all $n \in \mathbb{N}$, because

$$x^*\ x \geq x\ x^* \geq x^2\ x^{2\,*} \geq x^3\ x^{3\,*} \geq ...,$$

as projections on H. Since $\pi(x,\, s)^* = \pi(s^*,\, x^*)$, for any $x,\, s \in \mathcal{G}_W$, Therefore, we can get the conditional glued graph $G = G_x \#_\pi G_s$, graph-isomorphic to the following graph,

$$\begin{array}{ccccccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \cdots \end{array}$$

The upper row of the above graph can be regarded as the graph G_x, and the columns are regarded as the graphs G_s's. This shows that, similar to the previous lemma, we can conclude that

$$C^*(\{x,\, s\}) \overset{*\text{-isomorphic}}{=} \overline{C^*(\{x\}) *_{a\lg} C^*(\{s\})}.$$

Lemma 4.7. *Let $x \in \mathcal{G}_f^{(s)}$, and $s \in \mathcal{G}_\infty^{(s)}$. Then the C^*-algebra $C^*(\{x,\, s\})$ is $*$-isomorphic to the C^*-closure $\overline{C^*(\{x\}) *_{a\lg} C^*(\{s\})}$ of the algebraic free product algebra of $C^*(\{x\})$ and $C^*(\{s\})$. Thus, $C^*(\{x,\, s\})$ is $*$-isomorphic to*

$$\overline{(\mathbb{C} \cdot 1_{H_{\{x,\, s\}}}) \otimes_{\mathbb{C}} (\mathcal{T}(H_x) \otimes_{\mathbb{C}} M_2(\mathbb{G}))}.$$

□

Finally, let's take $x,\, y \in \mathcal{G}_f^{(s)}$, and assume that $\pi(x,\, y) \neq 0$, in $\mathbb{G}_0$. Let $H_{\{x,\, y\}}$ be the $\{x,\, y\}$-Hilbert space in the sense of Section 3, induced by the subspaces H_x and H_y of H, and let $\alpha \overset{denote}{=} \alpha\mid_{\mathbb{G}_{x,\, y}}$ be the restriction of the $\mathcal{G}$-groupoid action α, where $\mathbb{G}_{x,\, y}$ is the graph groupoid, which is a subgroupoid of the $\mathcal{G}$-groupoid $\mathbb{G}$, induced by the conditional glued graph $G_{x,\, y} = G_x \#_\pi G_y$ of G_x and G_y of $\mathcal{G}_G$. We know that

$$C^*(\{x,\, y\}) \overset{*\text{-isomorphic}}{=} C^*_\alpha(\mathbb{G}_{x,\, y}),$$

$$\overset{*\text{-isomorphic}}{=} C^*_\alpha(\mathbb{G}_x) *^r_{top} C^*_\alpha(\mathbb{G}_y)$$

by the definition of $\mathbb{G}_{x,\, y}$ (or $G_{x,\, y}$) and by Section 4.2

$$\overset{*\text{-isomorphic}}{=} \overline{C^*_\alpha\,(\mathbb{G}_x \;\; *^r \;\; \mathbb{G}_y)}$$

$$\overset{*\text{-isomorphic}}{=} \overline{C^*(\{x\}) *^r_{a\lg} \;\; C^*(\{y\})} \qquad \left(\overset{*\text{-isomorphic}}{\neq} \overline{C^*(\{x\}) *_{a\lg} C^*(y)} \right)$$

$$\overset{*\text{-isomorphic}}{=} \mathcal{T}(H_x) *^r_{top} \mathcal{T}(H_y), \tag{4.8}$$

as a C^*-subalgebra of $B(H)$. By the reduction on "$*^r_{top}$" (and on "$*^r_{a\lg}$" and on "$*^r$"), it is somewhat hard to deal with it, compared with the previous cases. In the previous cases, we only need to observe the non-reduced free products, moreover simple algebraic (non-reduced) free products. However, in the above case, we need to observe the reduction on the free product.

In fact, it is also interesting to observe the $*$-isomorphic C^*-subalgebra of $\mathcal{T}(H_x) *^r_{top} \mathcal{T}(H_y)$, in (4.8), inside $B(H)$. Clearly, if the corresponding graphs G_x and G_y in $\mathcal{G}_G$ satisfies either $G_x > G_y$ or $G_x < G_y$, or equivalently, if either

$$\dim(\ker y^*) \mid \dim(\ker x^*) \text{ or } \dim(\ker x^*) \mid \dim(\ker y^*),$$

then $G_{x,\,y} = G_y$, respectively, $G_{x,\,y} = G_x$. Therefore, we can get that:

Lemma 4.8. *Let $x,\, y \in \mathcal{G}_f^{(s)}$, and assume that the corresponding graphs G_x and G_y in $\mathcal{G}_G$ satisfy $G_x > G_y$. Then C^*-algebra $C^*(\{x,\, y\})$ generated by $\{x,\, y\}$ is $*$-isomorphic to the Toepilitz algebra $\mathcal{T}(H_y)$. i.e.,*

$$C^*(\{x,\, y\}) \overset{*\text{-isomorphic}}{=} C^*_\alpha(\mathbb{G}_y) \overset{*\text{-isomorphic}}{=} \mathcal{T}(H_y).$$

□

Assume now that $\pi(x,\, y) \neq 0$ in $\mathbb{G}_0$, and also assume that neither $G_x > G_y$ nor $G_x < G_y$. Then the conditional glued graph $G_{x,\,y}$ of G_x and G_y is graph-isomorphic to the following graph,

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \cdots
\end{array}$$

We can regard all edges of the previous graph as shifts x's or y's. Thus all finite paths of the above graph are regarded as shifts, too. And hence, all reduced finite paths in the graph groupoid $\mathbb{G}_{x,\,y}$ of $G_{x,\,y}$ are regarded as finite dimensional shifts on H. So, it is easy to check that

$$\mathcal{T}(H_x) *^r_{top} \mathcal{T}(H_y) \overset{C^*\text{-subalgebra}}{\subseteq} \mathcal{T}(H_{\{x,\,y\}}), \tag{4.9}$$

in $B(H)$. Conversely, assume that $T \in \mathcal{T}(H_{\{x,\,y\}})$. Since $\mathcal{T}(H_{\{x,\,y\}})$ is $*$-isomorphic to

$$C_{\alpha}^{*}(\mathbb{G}_{x,\,y}) \overset{*\text{-isomorphic}}{=} C_{\alpha}^{*}(\mathbb{G}_{x}) *_{top}^{r} C_{\alpha}^{*}(\mathbb{G}_{y}) \overset{*\text{-isomorphic}}{=} \overline{\mathbb{C}[\mathbb{G}_{x} *^{r} \mathbb{G}_{y}]},$$

the element T is represented by

$$T = \sum_{w \in \mathbb{G}_{x,\,y}} t_{w}\ w, \text{ for } t_{w} \in \mathbb{C}.$$

Since $\mathbb{G}_{x,\,y} = \mathbb{G}_{x} *^{r} \mathbb{G}_{y}$, the operator T is contained in $\overline{\mathbb{C}[\mathbb{G}_{x,\,y}]}$ in $B(H_{\{x,\,y\}}) \subseteq B(H)$. This shows that

$$\mathcal{T}(H_{\{x,\,y\}}) \overset{C^{*}\text{-subalgebra}}{\subseteq} \mathcal{T}(H_{x}) *_{top}^{r} \mathcal{T}(H_{y}), \tag{4.10}$$

inside $B(H)$, whenever x and y are finite dimensional shifts satisfying that $\pi(x, y) \neq 0$ and neither $G_{x} < G_{y}$ nor $G_{x} < G_{y}$.

By (4.9) and (4.10), we can get the following lemma.

Lemma 4.9. *Let $x,\, y \in \mathcal{G}_{f}^{(s)}$ and assume that $\pi(x,\, y) \neq 0$ in $\mathbb{G}_{0}$, and neither $G_{x} < G_{y}$ nor $G_{y} < G_{x}$ in $\mathcal{G}_{G}$. Then the C^{*}-algebra $C^{*}(\{x,\, y\})$ is $*$-isomorphic to the Toeplitz algebra $\mathcal{T}(H_{\{x,\,y\}})$, where $H_{\{x,\,y\}}$ is the $\{x,\, y\}$-Hilbert space in the sense of Section 3.* $\square$

Readers can realize the difference between the above lemma and the previous lemmas. The above lemma shows that, inductively, if we have a subfamily $\mathcal{G}_{f}$ of $\mathcal{G}_{f}^{(s)}$, satisfying that (i) $\pi(x_{1},\, x_{2}) \neq 0$ in $\mathbb{G}_{0}$, and (ii) neither $G_{x_{1}} < G_{x_{2}}$ nor $G_{x_{2}} < G_{x_{1}}$, in $\mathcal{G}_{G}$, for all pair $(x_{1},\, x_{2}) \in \mathcal{G}_{f} \times \mathcal{G}_{f}$, then the C^{*}-subalgebra $C^{*}(\mathcal{G}_{f})$ of $B(H)$, generated by $\mathcal{G}_{f}$, is $*$-isomorphic to the Toeplitz algebra $\mathcal{T}(H_{\mathcal{G}_{f}})$, where $H_{\mathcal{G}_{f}}$ is the $\mathcal{G}_{f}$-Hilbert space in the sense of Section 3, which is a subspace of $H_{\mathcal{G}} \subseteq H$.

Proposition 4.10. *Let $\mathcal{G}_{f}$ be a subfamily of $\mathcal{G}_{f}^{(s)} \subseteq \mathcal{G}_{W}$ satisfying (i) $\pi(x_{1},\, x_{2}) \neq 0$ in $\mathbb{G}_{0}$, and (ii) neither $G_{x_{1}} < G_{x_{2}}$ nor $G_{x_{1}} > G_{x_{2}}$ in $\mathcal{G}_{G}$, for all $(x_{1},\, x_{2}) \in \mathcal{G}_{f} \times \mathcal{G}_{f}$, then the C^{*}-subalgebra $C^{*}(\mathcal{G}_{f})$ of $B(H)$ is $*$-isomorphic to $\mathcal{T}(H_{\mathcal{G}_{f}})$, where $H_{\mathcal{G}_{f}}$ is the $\mathcal{G}_{f}$-Hilbert space, which is a subspace of $H_{\mathcal{G}}$.* $\square$

Let $\mathcal{G}_{f}^{(s)} \subset \mathcal{G}_{W}$ be nonempty. Define the corresponding subset $\mathcal{G}_{G}^{(s:f)}$ of $\mathcal{G}_{G}$ by

$$\mathcal{G}_{G}^{(s:f)} \overset{def}{=} \{G_{x} \in \mathcal{G}_{G} : x \in \mathcal{G}_{f}^{(s)}\} \subseteq \mathcal{G}_{G}.$$

Recall that the family $\mathcal{G}_{G}$ is in fact a partially ordered set $(\mathcal{G}_{G},\, \leq)$, where "$\leq$" is the full-subgraph inclusion on directed graphs. Then, without loss of generality, we can regard $\mathcal{G}_{G}^{(s:f)}$ as a partially ordered set $(\mathcal{G}_{G}^{(s:f)},\, \leq)$, under the inherited partial

ordering $\leq$. By this partial ordering, we can take the chains inside $\mathcal{G}_G^{(s:f)}$, and, for each chain, we can take the minimal elements in $\mathcal{G}_G^{(s:f)}$.

Definition 4.5. *Let $\mathcal{G}_G^{(s:f)} \subseteq \mathcal{G}_G$ be given as before. Define the subset $\mathcal{G}_G^{(s:\min)}$ of $\mathcal{G}_G^{(s:f)} \subseteq \mathcal{G}_G$ by*

$$\mathcal{G}_G^{(s:\min)} \overset{def}{=} \left\{ G \in \mathcal{G}_G^{(s:f)} : G \text{ is minimal in } \mathcal{G}_G^{(s:f)} \right\}.$$

By the collection $\mathcal{G}_G^{(s:\min)}$, we can take the corresponding subset $\mathcal{G}_f^{(s:\min)}$ of $\mathcal{G}_f^{(s)}$ by

$$\mathcal{G}_f^{(s:\min)} \overset{def}{=} \left\{ x \in \mathcal{G}_f^{(s)} : G_x \in \mathcal{G}_G^{(s:\min)} \right\}.$$

By the previous lemmas, we can realize that:

Proposition 4.11. *The C^*-subalgebra $C^*\left(\mathcal{G}_f^{(s)}\right)$ of $C^*(\mathcal{G})$ is $*$-isomorphic to*

$$\left(\mathbb{C} \cdot 1_{H_{\mathcal{G}_f^{(s)}}}\right) \otimes_{\mathbb{C}} C_\alpha^*\left(\mathbb{G}_f^{(s:\min)}\right), \tag{4.11}$$

where $\mathbb{G}_f^{(s:\min)}$ is the subgroupoid of the $\mathcal{G}$-groupoid $\mathbb{G}$, generated by the graph $G_f^{(s:\min)} \overset{def}{=} \underset{x \in \mathcal{G}_f^{(s:\max)}}{\#_\pi} G_x$. $\square$

Observe the graph $G_f^{(s:\min)} \overset{def}{=} \underset{x \in \mathcal{G}_f^{(s:\max)}}{\#_\pi} G_x$. By the previous lemmas, we can conclude that if $x,\, y \in \mathcal{G}_f^{(s:\min)}$ having their corresponding graphs $G_x,\, G_y \in \mathcal{G}_G^{(s:\min)}$, then there are two cases where $\pi(x,\, y) \neq 0$ or $\pi(x,\, y) = 0$, in $\mathbb{G}_0$. In particular, if $\pi(x,\, y) \neq 0$, the full-subgraph $G_{x,\, y} = G_x \#_\pi G_y$ of $G_f^{(s:\min)}$ induces the graph groupoid $\mathbb{G}_{x,\, y} = \mathbb{G}_x *^r \mathbb{G}_y$, and $C_\alpha^*(\mathbb{G}_{x,\, y})$ is $*$-isomorphic to $\mathcal{T}(H_{\{x,\, y\}})$. Clearly, if $\pi(x,\, y) = 0$, then

$$C_\alpha^*(\mathbb{G}_{x,\, y}) \overset{*\text{-isomorphic}}{=} C_\alpha^*(\mathbb{G}_x) \oplus C_\alpha^*(\mathbb{G}_y) \overset{*\text{-isomorphic}}{=} \mathcal{T}(H_x) \oplus \mathcal{T}(H_y).$$

Therefore, we can get that:

Proposition 4.12. *Let $\mathcal{G}_f^{(s:\min)}$ of $\mathcal{G}_W$ be given as above. Then there exists $k \in \mathbb{N}$ such that (i) $k \leq \left|\mathcal{G}_f^{(s)}\right|$, and (ii) we can decompose $\mathcal{G}_f^{(s:\min)}$ by $\mathcal{G}_f^{(s:\min)} = \overset{k}{\underset{m=1}{\sqcup}} \mathcal{G}_{f,\, m}^{(s:\min)}$, where $\sqcup$ means the disjoint union and where*

$$x_1 \neq x_2 \in \mathcal{G}_{f,\,m}^{(s:\min)} \Longleftrightarrow \pi(x_1,\, x_2) \neq 0 \;\; or\; \pi(x_2,\, x_1) \neq 0 \;\; in\; \mathbb{G}_0.$$

So, the C^*-algebra $C^*\left(\mathcal{G}_f^{(s:\min)}\right)$ is $*$-isomorphic to $\underset{m=1}{\overset{k}{\oplus}} \mathcal{T}\left(H_{\mathcal{G}_{f,m}^{(s:\min)}}\right)$, as C^*-subalgebras of $B(H_{\mathcal{G}}) \subseteq B(H)$. $\square$

The proof of the previous proposition is done by the previous lemmas and proposition, by induction.

Notation In the rest of this paper, we will keep using the same notations used in the previous lemmas and propositions. $\square$

Now, we will combine all previous observations. The following theorem provides the topological free block structures in $C^*(\mathcal{G})$.

Theorem 4.13. *Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$. Define the set $\mathcal{G}_W^{\min}$ of partial isometries induced by $\mathcal{G}_W$ of $\mathcal{G}$ by*

$$\mathcal{G}_W^{\min} = \mathcal{G}^{(u)} \cup \mathcal{G}_\infty^{(s)} \cup \mathcal{G}_f^{(s:\min)} = \mathcal{G}^{(u)} \cup \mathcal{G}_\infty^{(s)} \cup \left(\underset{m=1}{\overset{k}{\sqcup}} \mathcal{G}_{f:m}^{(s:\min)} \right).$$

Then the C^*-subalgebra $C^*(\mathcal{G})$ of $B(H)$ is $*$-isomorphic to

$$(\mathbb{C} \cdot 1_{H_{\mathcal{G}}}) \otimes_{\mathbb{C}} C^*\left(\mathcal{G}_W^{\min}\right),$$

and hence it is $*$-isomorphic to

$$\overline{\left(\underset{u \in \mathcal{G}^{(u)}}{*_{a\lg}} \left((\mathbb{C} \cdot 1_{H_u}) \otimes_{\mathbb{C}} C(spec(u)) \right) \right)}$$

$$\overline{*_{a\lg} \left(\underset{s \in \mathcal{G}_\infty^{(s)}}{*_{a\lg}} \left((\mathbb{C} \cdot 1_{H_s}) \otimes_{\mathbb{C}} M_2(\mathbb{C}) \right) \right) *_{a\lg} \left(\underset{m=1}{\overset{k}{\oplus}} \mathcal{T}_m \right)},$$

where $\mathcal{T}_m = \mathcal{T}\left(H_{\mathcal{G}_{f,\,m}^{(s:\min)}}\right)$ are the Toeplitz algebras, for all $m = 1,\, \ldots,\, k$.

Proof. Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$ and let $\mathcal{G}_W^{\min}$ be given as above. Then

$$C^*(\mathcal{G}) \overset{*\text{-isomorphic}}{=} C_\alpha^*(\mathbb{G}) \overset{*\text{-isomorphic}}{=} (\mathbb{C} \cdot 1_{H_{\mathcal{G}}}) \otimes_{\mathbb{C}} C^*\left(\mathcal{G}_W^{\min}\right)$$

by the previous lemmas and propositions

$$\overset{*\text{-isomorphic}}{=} (\mathbb{C} \cdot 1_{H_{\mathcal{G}}}) \otimes_{\mathbb{C}} \left(C^{*}\left(\mathcal{G}^{(u)}\right) *_{top}^{r} C^{*}\left(\mathcal{G}_{\infty}^{(s)}\right) *_{top}^{r} C^{*}\left(\underset{m=1}{\overset{k}{\sqcup}} \mathcal{G}_{f,m}^{(s:\min)} \right) \right)$$

$$\overset{*\text{-isomorphic}}{=} \left(\underset{u \in \mathcal{G}^{(u)}}{*_{top}^{r}} \mathcal{A}_{u} \right) *_{top}^{r} \left(\underset{s \in \mathcal{G}_{\infty}^{(s)}}{*_{top}^{r}} \mathcal{A}_{s} \right) *_{top}^{r} \left(\underset{m=1}{\overset{k}{\oplus}} \mathcal{T}_{m} \right)$$

where $\mathcal{A}_{x} = C^{*}(\{x\})$, for all $x \in \mathcal{G}_{W}$, and $\mathcal{T}_{m} = \mathcal{T}\left(H_{\mathcal{G}_{f,\,m}^{(s:\min)}}\right)$, by the previous propositions, for all $m = 1, \ldots, k$

$$\overset{*\text{-isomorphic}}{=} \left(\overline{\underset{u \in \mathcal{G}^{(u)}}{*_{a\lg}} \mathcal{A}_{u}} \right) *_{top}^{r} \left(\overline{\underset{s \in \mathcal{G}_{\infty}^{(s)}}{*_{a\lg}} \mathcal{A}_{s}} \right) *_{top}^{r} \left(\underset{m=1}{\overset{k}{\oplus}} \mathcal{T}_{m} \right)$$

$$\overset{*\text{-isomorphic}}{=} \overline{\left(\underset{u \in \mathcal{G}^{(u)}}{*_{a\lg}} \mathcal{A}_{u} \right) *_{a\lg} \left(\underset{s \in \mathcal{G}_{\infty}^{(s)}}{*_{a\lg}} \mathcal{A}_{s} \right) *_{a\lg} \left(\underset{m=1}{\overset{k}{\oplus}} \mathcal{T}_{m} \right)},$$

by the previous lemmas. Recall that

$$\mathcal{A}_{u} \overset{*\text{-isomorphic}}{=} (\mathbb{C} \cdot 1_{H_{u}}) \otimes_{\mathbb{C}} C\left(spec(u)\right), \text{ for all } u \in \mathcal{G}^{(u)},$$

and

$$\mathcal{A}_{s} \overset{*\text{-isomorphic}}{=} (\mathbb{C} \cdot 1_{H_{s}}) \otimes_{\mathbb{C}} M_{2}(\mathbb{C}), \text{ for all } s \in \mathcal{G}_{\infty}^{(s)}.$$

∎

The above theorem provides the characterization of the block structures of C^{*}-subalgebras generated by finitely many partial isometries in terms of the algebraic (non-reduced) free product and the characterization of C^{*}-subalgebras generated by a single partial isometries.

4.3. **Examples.**

In this section, we will consider some examples.

Example 4.1. *In this first example, we will consider the graph-families we observed in [13]. Let $\mathcal{G} = \{a_{1}, \ldots, a_{N}\}$ be a finite family of partial isometries in $B(H)$. And assume that $\mathcal{G}$ constructs a finite directed graph G in the sense that*

(i) $|\mathcal{G}| = |E(G)|$ *and* $|\mathcal{G}_{p}| = |V(G)|$, *where*

$$\mathcal{G}_{p} \overset{def}{=} \{a_{j}^{*}\, a_{j} : j = 1, \ldots, N\} \cup \{a_{j}\, a_{j}^{*} : j = 1, \ldots, N\},$$

with the corresponding bijection $h_E : E(G) \rightarrow \mathcal{G}$.

(ii) the edges e_1 *and* e_2 *generate nonempty finite path* $e_1\ e_2$ *if and only if* $H_{init}^{h_E(e_1)} = H_{fin}^{h_E(e_2)}$, *where the symbol "*$H_1 = H_2$*" means that* H_1 *and* H_2 *are identically same as subspaces of* H. *So, "*$H_1 \neq H_2$*" means that* $H_1 \cap H_2 = \{0_H\}$.

The above graph-family-setting makes us understand the elements a_j *satisfies that either* $a_j = u_j$ *or* $a_j = s_j$, *where* u_j *are the unitary parts of* a_j *and* s_j *are the shift parts of* a_j. *Moreover, by the condition (ii), if* $a_j = s_j$, *for some* $j \in \{1, ..., N\}$, *then its* $*$*-isomorphic index* $i_*(a_j) = (0, \varepsilon, \infty, 0)$, *for some* $\varepsilon \in \mathbb{N}_0^\infty$. *Thus we can get that the* C^**-subalgebra* $C^*(\{a_j\})$ *generated by a single partial isometry* $a_j \in \mathcal{G}$ *is* $*$*-isomorphic to either*

$$(\mathbb{C} \cdot 1_{H_{a_j}}) \otimes_{\mathbb{C}} C(\mathbb{T})$$

(if $h_E^{-1}(a_j)$ *is a loop-edge in* $E(G)$*) or*

$$(\mathbb{C} \cdot 1_{H_{a_j}}) \otimes_{\mathbb{C}} M_2(\mathbb{C})$$

(if $h_E^{-1}(a_j)$ *is a non-loop edge in* $E(G)$*). i.e., the* $\mathcal{G}_W$*-admissibility is completely determined by (ii), and hence, we have either,*

$$\pi(a_i, a_j) = 0$$

or

$$\pi(a_i, a_j) = (a_i^*\ a_i)\ (a_j\ a_j^*) = a_i^*\ a_i = a_j\ a_j^*,$$

for all $i, j \in \{1, ..., N\}$. *In other words, in this case, we have that*

$$\mathcal{G} = \mathcal{G}_W = \mathcal{G}^{(u)} \cup \mathcal{G}_\infty^{(s)}, \text{ with } \mathcal{G}_f^{(s)} = \varnothing.$$

Therefore, indeed, the C^**-algebras* $C^*(\{a_j\})$ *are* $*$*-isomorphic to either*

$$(\mathbb{C} \cdot 1_{H_0}) \otimes_{\mathbb{C}} C(\mathbb{T}) \text{ or } (\mathbb{C} \cdot 1_{H_0}) \otimes_{\mathbb{C}} M_2(\mathbb{C}),$$

by Section 3.1. Moreover, if G *is connected in the sense that, for any* $(v_1, v_2) \in V(\widehat{G}) \times V(\widehat{G})$, *there always exists at least one reduced finite path* $w \in FP_r(\widehat{G})$ *such that* $w = v_1\ w\ v_2$ *and* $w^{-1} = v_2\ w^{-1}\, v_1$, *then the* C^**-subalgebra* $C^*(\mathcal{G})$ *generated by* $\mathcal{G}$ *is* $*$*-isomorphic to the affiliated matricial graph* C^**-algebra* $\mathcal{M}_G(H_0)$ *which is* $*$*-isomorphic to*

$$(\mathbb{C} \cdot 1_{H_0}) \otimes_{\mathbb{C}} \mathcal{M}_G,$$

where $\mathcal{M}_G$ is the matricial graph C^*-algebra which is the C^*-subalgebra of $M_n(\mathbb{C})$ (See [13]), and where H_0 is the Hilbert space that is Hilbert-space isomorphic to the initial and the final subspaces of all elements in $\mathcal{G}$. Notice that, indeed, $H_0 = H_{\mathcal{G}}$ in H.

So, in general, if the family $\mathcal{G}$ constructs a finite directed graph G in the above sense and if G has its connected components $G_1, \ldots, G_t$, for some $t \in \mathbb{N}$, where $G_1, \ldots, G_t$ are the connected full-subgraphs of G. Then we can decompose $\mathcal{G}$ into the disjoint union of $\mathcal{G}_1, \ldots, \mathcal{G}_t$, constructing $G_1, \ldots, G_t$, respectively. Then the C^*-algebra $C^*(\mathcal{G})$ is $\oplus_{j=1}^t C^*(\mathcal{G}_j)$ which is $*$-isomorphic to the direct sum of the affiliated matricial graph C^*-algebras:

$$C^*(\mathcal{G}) = \oplus_{j=1}^t C^*(\mathcal{G}_t) \overset{*\text{-isomorphic}}{=} \oplus_{j=1}^t \mathcal{M}_G(H_j),$$

where H_j's are the embedded subspaces of H which are Hilbert-space isomorphic to $p_j\ H$, for all $p_j \in C^*(\mathcal{G}_p^{(j)})$, for all $j = 1, \ldots, t$.

Example 4.2. *Let U be the unilateral shift on the Hilbert space $H = l^2(\mathbb{N}_0)$, and let $k_1 > k_2 \in \mathbb{N}$. We can have the shift operators U^{k_1} and U^{k_2} on H. Then they generate the C^*-subalgebras $C^*(\{U^{k_1}\})$ and $C^*(\{U^{k_2}\})$ of $B(H)$, which are $*$-isomorphic to the classical Toeplitz algebra $\mathcal{T}(H)$. By hypothesis, $\pi(U^{k_1}, U^{k_2}) \neq 0$ on H. Therefore, $C^*\left(\{U^{k_1}, U^{k_2}\}\right)$ is $*$-isomorphic to the classical Toeplitz algebra $\mathcal{T}(H)$, by Section 4.2, since $H_{\{U^{k_1}, U^{k_2}\}}$ is Hilbert-space isomorphic to $H = l^2(\mathbb{N}_0)$. Indeed, if $\mathcal{G}_W = \{U^{k_1}, U^{k_2}\}$, then $\mathcal{G}_f^{(s:\max)} = \{U^{k_2}\}$. Therefore, we can get the desired result.*

Example 4.3. *Let $k \in \mathbb{N}$ and U, the unilateral shift on*

$$H = l^2(\mathbb{N}_0) \oplus l^2(\mathbb{N}_0) \overset{denote}{=} H_1 \oplus H_2.$$

Define the operator V on H by

$$V \overset{def}{=} \begin{pmatrix} 0_H & 0_H \\ 1_H & 0_H \end{pmatrix} \text{ on } \begin{array}{c} H_1 \\ \oplus \\ H_2 \end{array}.$$

Consider the family $\mathcal{G} = \{x_1, x_2\}$, where $x_1 = U^k$ and $x_2 = V$. We can check that

$$i_*(x_1) = (0, 0, k, 0) \text{ and } i_*(0, 0, \infty, 0) \text{ in } (\mathbb{N}_0^\infty)^4.$$

Therefore, the C^*-subalgebras $\mathcal{A}_{x_k} = C^*(\{x_k\})$ of $B(H)$, for $k = 1, 2$, satisfy that

$$\mathcal{A}_{x_1} \overset{*\text{-isomorphic}}{=} \mathcal{T}(H),$$

and

$$\mathcal{A}_{x_2} \overset{*\text{-isomorphic}}{=} (\mathbb{C} \cdot 1_{H_1}) \otimes_{\mathbb{C}} M_2(\mathbb{C}),$$

by Section 3.1. Also, we can have the family $\mathcal{G}_G = \{G_{x_1}, G_{x_2}\}$ *of the corresponding graphs* G_{x_k} *of* x_k*, for* $k = 1, 2$*. The graphs are*

$$G_{x_1} = \underset{x_1^* x_1}{\bullet} \xrightarrow{x_1} \underset{x_1 x_1^*}{\bullet} \xrightarrow{x_1^{(2)}} \underset{x_1^2 x_1^{2*}}{\bullet} \xrightarrow{x_1^{(3)}} \underset{x_1^3 x_1^{3*}}{\bullet} \xrightarrow{x_1^{(4)}} \underset{x_1^4 x_1^{4*}}{\bullet} \xrightarrow{x_k^{(5)}}$$

and

$$G_{x_2} = \underset{x_2^* x_2}{\bullet} \xrightarrow{x_2} \underset{x_2 x_2^*}{\bullet}.$$

We can check that

$$\pi(x_1^n, x_2^m) = (x_1^{n\,*}\ x_1^n)\ (x_2^m\ x_2^{m\,*}) = \begin{cases} x_2^m\ x_2^{m\,*} \neq 0 & \text{if } m = 1 \\ 0 & \text{if } m \neq 1, \end{cases}$$

since $x_2^k = 0$*, for all* $k \in \mathbb{N} \setminus \{1\}$*, and*

$$\pi(x_1^{n\,*}, x_2^{m\,*}) = (x_1^n\ x_1^{n\,*})\ (x_2^{m\,*}\ x_2^m) = \begin{cases} (1_H)(1_{H_1} \oplus 0) = 1_{H_1} & \text{if } m = 1 \\ 0 & \text{if } m \neq 1, \end{cases}$$

since $x_2^k = 0$*, for all* $k \in \mathbb{N} \setminus \{1\}$*, for all* $n, m \in \mathbb{N}$*. Thus we can get the conditional iterated glued graph* $G = G_{x_1} \#_\pi G_{x_2}$ *as the graph which is graph-isomorphic to the following graph:*

$$G = \begin{array}{ccccccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \cdots \end{array}$$

By Section 4.2, the C^**-algebra* $\mathcal{A} = C^*_\alpha(\mathbb{G})$ *is* $*$*-isomorphic to*

$$\mathcal{A} \overset{*\text{-isomorphic}}{=} \overline{(\mathcal{T}(H)) *_{a\lg} ((\mathbb{C} \cdot 1_{H_1}) \otimes_{\mathbb{C}}\ M_2(\mathbb{C}))}.$$

Example 4.4. *Let* $H \overset{Hilbert\text{-}Space}{=} l^2(\mathbb{N}_0)$ *having its Hilbert basis* $\mathcal{B}_H = \{\xi_n : n \in \mathbb{N}_0\}$*, where*

$$\xi_0 = (1, 0, 0, 0, ...)$$

and

$$\xi_n = \left(\underbrace{0,\ ...,\ 0}_{n\text{-}times},\ 1,\ 0,\ 0,\ ...\right), \textit{ for all } n \in \mathbb{N}.$$

Define an operator $x \in B(H)$ *by the operator having its infinite matricial form,*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \boxed{1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & \cdots \\ & & & \vdots & & & \ddots \\ & & & & & & \end{pmatrix}.$$

i.e., the operator x *satisfies that*

$$(t_0,\ t_1,\ t_2,\ t_3,\ ....) \overset{x}{\longmapsto} (0,\ t_0,\ 0,\ t_1,\ 0,\ t_2,\ 0,\ t_3,\ 0,\ ...),$$

for all $(t_0,\ t_1,\ t_2,\ ...) \in H$. *The adjoint* x^* *of* x *has its infinite matricial form,*

$$\begin{pmatrix} 0 & 0 & \boxed{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & & & \vdots & & & \ddots \\ & & & & & & \end{pmatrix},$$

so, it satisfies that

$$(t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ ...) \overset{x^*}{\longmapsto} (t_1,\ t_3,\ t_5,\ ...),$$

for all $(t_0,\ t_1,\ t_2,\ ...) \in H$.

Instead of observing the Wold decomposition of x, *we will consider* x *by the operator determined by the (finite dimensional) shifts* $(y_n)_{n=0}^{\infty}$. *First, we define the following subspaces* K_n*'s of* H*:*

$$K_n \overset{def}{=} \mathbb{C} \cdot \xi_n \overset{\text{Hilbert-Space}}{=} \mathbb{C} \overset{\text{Subspace}}{\subset} H, \textit{ for all } n \in \mathbb{N}_0.$$

Now, define the operators y_n *by*

$$y_0 : (t,\ 0,\ 0,\ 0,\ ...) \longmapsto (0,\ t,\ 0,\ 0,\ 0,\ ...)$$

and

$$y_n : \left(\underbrace{0,\ ....,\ 0}_{n\text{-}times},\ t,\ 0,\ 0,\ ... \right) \mapsto \left(\underbrace{0,\ ....,\ 0}_{(2n+1)\text{-}times},\ t,\ 0,\ 0,\ ... \right)$$

on H, for all $n \in \mathbb{N}$ and $t \in \mathbb{C}$. Then each operator y_n is a partial isometry on H having its initial space and its final space as follows:

$$H_{init}^{y_0} = K_0 \text{ and } H_{fin}^{y_0} = K_1$$

and

$$H_{init}^{y_n} = K_n \text{ and } H_{fin}^{y_n} = K_{2n+1}, \text{ for all } n \in \mathbb{N}.$$

Then the given operator x is defined by

$$x\ \xi_k \overset{def}{=} y_k\ \xi_k, \text{ for all } k \in \mathbb{N}_0,\ \xi_k \in \mathcal{B}_H.$$

i.e., we can define

$$x \left(\sum_{k=0}^{\infty} t_k\ \xi_k \right) \overset{def}{=} \sum_{k=0}^{\infty} t_k\ y_k(\xi_k) = \sum_{k=0}^{\infty} t_k\ \xi_{2k+1},$$

for all $\sum_{k=0}^{\infty} t_k\ \xi_k \in H$, where $t_k \in \mathbb{C}$. Now, construct the subspaces $(H_n)_{n=0}^{\infty}$ of H by

$$H_n = K_n \oplus K_{2n+1}, \text{ for all } n \in \mathbb{N}_0.$$

Note that $H_n \overset{Hilbert\text{-}Space}{=} \mathbb{C}^{\oplus 2}$, for all $n \in \mathbb{N}_0$. Then the operators y_n can be understood as shifts on H_n. (Notice that they are regarded as shifts on finite dimensional space $\mathbb{C}^{\oplus 2}$.) Moreover, the operators y_n on H_n have their matricial forms,

$$y_n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ on } H_n = \begin{array}{c} K_n \\ \oplus \\ K_{2n+1} \end{array},$$

for all $n \in \mathbb{N}_0$. Therefore, the C^*-algebras $C^*(\{y_n\})$ generated by the shifts y_n are $*$-isomorphic to $M_2(\mathbb{C}) = B(H_n)$, as embedded C^*-subalgebras of $B(H)$, for all $n \in \mathbb{N}_0$. Similar to Section 3.3, if we let $\mathcal{G} = \{y_n \in B(H) : n \in \mathbb{N}_0\}$ (by regarding y_n's as operators on H), then we get the $\mathcal{G}$-space $H_{\mathcal{G}} = H$, and we can have the $\mathcal{G}$-action α of the $\mathcal{G}$-groupoid $\mathbb{G}$, where

$$G_{y_n} = \underset{y_n^* y_n}{\bullet} \overset{y}{\longrightarrow} \underset{y_n y_n^*}{\bullet} \in \mathcal{G}_G, \quad \text{for all } n \in \mathbb{N}_0.$$

Define a map $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ *by*

$$f(m) = 2m + 1, \text{ for all } m \in \mathbb{N}_0.$$

We will denote the iterated compositions $\underbrace{f \circ ...... \circ f}_{k\text{-times}}$ *by* $f^{(k)}$, *for all* $k \in \mathbb{N}$. *Then we can have the subset* $X_{(n)}$ *of* $\mathbb{N}$ *by*

$$X_{(n)} \overset{def}{=} \{n\} \cup \{f^{(k)}(n) : k \in \mathbb{N}\} \subset \mathbb{N}_0,$$

for all $n \in \mathbb{N}_0$. *It is easy to check that if* $n \in \mathbb{N}_0$, *then, for any* $k \in \mathbb{N}$, *the sets* $X_{(f^{(k)}(n))}$ *are contained in* $X_{(n)}$. *i.e.,*

$$X_{(f^{(k)}(n))} \subset X_{(n)}, \text{ for all } n \in \mathbb{N}_0 \text{ and } k \in \mathbb{N}.$$

For instance, $X_{(7)} \subset X_{(3)} \subset X_{(1)} \subset X_{(0)}$, *etc. This means that we can take chains under the usual set-inclusion on the collection* $\mathcal{Y} = \{X_{(n)} : n \in \mathbb{N}_0\}$ *of* $X_{(n)}$*'s. Denote* $\mathcal{X}$ *by the collection of all maximal elements of* $\mathcal{Y}$, *under the partial ordering* $\subset$. *For instance,* $X_{(0)}, X_{(2)}, X_{(4)}, X_{(6)}$ *are the first five elements of* $\mathcal{X}$. *Let* I *be the subset of* $\mathbb{N}_0$ *defined by*

$$I = \{n \in \mathbb{N}_0 : X_{(n)} \in \mathcal{X}\} \subset \mathbb{N}_0.$$

Let $\mathcal{G} = \{y_0, y_1, y_2, y_3, ...\}$ *and let* $\mathcal{G}_G = \{G_{(n)} : n \in I\}$. *Then we can define the conditional iterated glued graph*

$$G = \underset{n \in I}{\#_\pi} G_{(n)},$$

where

$$\begin{aligned} G_{(n)} &= G_n \; \#_\pi \left(\underset{k=n}{\overset{\infty}{\#_\pi}} G_{y_{2k+1}} \right) \\ &= \underset{y_n^* y_n}{\bullet} \overset{y_n}{\longrightarrow} \underset{y_n y_n^* = y_{2n+1}^* y_{2n+1}}{\bullet} \overset{y_{2n+1}}{\longrightarrow} \bullet \overset{y_{2(2n+1)+1}}{\longrightarrow} \bullet \cdots, \end{aligned}$$

for all $n \in I$. *Clearly, we can create the corresponding graph groupoid* $\mathbb{G}$ *of* G. *Then we can determine* $H_{\mathcal{G}} = H$ *and the groupoid action* α *of* $\mathbb{G}$ *on* H, *like Section 3.3. We can check that*

$$C^*(\mathcal{G}) \overset{*\text{-isomorphic}}{=} C^*_\alpha(\mathbb{G}),$$

by Section 3.4. By Section 4, we can check that the graph groupoid $\mathbb{G}_{(n)}$ *of* $G_{(n)}$ *generates the* C^**-algebra* $C^*_\alpha(\mathbb{G}_{(n)})$ *satisfies that*

$$C_{\alpha}^{*}(\mathbb{G}_{(n)}) \overset{*\text{-}isomorphic}{=} \underset{k=1}{\overset{\infty}{*_{top}^{r}}} \mathcal{M}_k, \text{ with } \mathcal{M}_k = B(\mathbb{C}\xi_k \oplus \mathbb{C}\xi_{2k+1}),$$

for all $k \in \mathbb{N}$, *and* $n \in I$. *Notice that* $\mathcal{M}_k \overset{*\text{-}isomorphic}{=} M_2(\mathbb{C})$, *for all* $k \in \mathbb{N}$.

5. Extensions of Unbounded Operators

In this Section, we will consider an application of our results to Unbounded Operator Theory, in particular, the Cayley Transform Theory. Let H be a countably infinite dimensional Hilbert space and let $\mathcal{D}$ be a dense subspace of H. A linear unbounded operator S defined on D is said to be *Hermitian* if

$$< S\,\xi_1,\, \xi_2 > = < \xi_1,\, S\,\xi_2 > \tag{6.1}$$

holds for all $\xi_1,\, \xi_2 \in \mathcal{D}$. Notice that, the relation (6.1) means that the dense subspace $\mathcal{D}$ is contained in the domain of the adjoint operator S^* of S, and that

$$S\,\xi = S^*\,\xi, \text{ for all } \xi \in \mathcal{D}.$$

In other words, the graph $\mathcal{G}(S)$ of S is contained in $\mathcal{G}(S^*)$ of S^*. Here, the graph $\mathcal{G}(S)$ of S means that the subset $\{(\xi,\, S\,\xi) : \xi \in \mathcal{D}\}$ of $H \times H$. We denote this relation by

$$S \subseteq S^*.$$

We are interested in finding Hermitian elements T of S; i.e., Hermitian operators T such that

$$\mathcal{G}(S) \subseteq \mathcal{G}(T). \tag{6.2}$$

Definition 5.1. *The Cayley transform* V *of* S *is a unbounded operator*

$$V \overset{def}{=} (S+i)\,(S-i)^{-1}. \tag{6.3}$$

i.e.,

$$V\,(S\,\xi - i\xi) = S\,\xi + i\xi, \text{ for all } \xi \in \mathcal{D}, \tag{6.4}$$

where i *of (6.3) means* $i1_H$, *with the imaginary number* i *satisfying* $i^2 = -1$ *in* $\mathbb{C}$, *where* 1_H *means the identity operator on* H.

It is immediate that the given unbounded operator S is Hermitian if and only if the Cayley transform V is a partial isometry. Moreover, the orthogonal compliment $\mathcal{E}_+$ of the operator $V^* V$ satisfies

(6.5)
$$\begin{aligned}\mathcal{E}_+ &= \ker(S^* - i) \\ &= \left\{\xi_+ \in dom(S^*) \,\middle|\, S^*\xi_+ = i\xi_+\right\},\end{aligned}$$

where $dom(T)$ means the domain of an operator T. Also, the orthogonal compliment $\mathcal{E}_-$ of $V\ V^*$ satisfies

(6.6)
$$\begin{aligned}\mathcal{E}_- &= \ker(S^* + i) \\ &= \left\{\xi_- \in dom(S^*) \,\middle|\, S^*\xi_- = -i\ \xi_-\right\}.\end{aligned}$$

Definition 5.2. *The subspaces $\mathcal{E}_+$ and $\mathcal{E}_-$ are called the defect spaces or the deficiency space of V. And the numbers*

$$n_+ \overset{def}{=} \dim \mathcal{E}_+ \ \text{and}\ n_- \overset{def}{=} \dim \mathcal{E}_- \tag{6.7}$$

are called the deficiency indices of V.

Lemma 5.1. *(See [48]) A Hermitian symmetric operator S has its self-adjoint extensions T if and only if $n_+ = n_-$. In this case, all self-adjoint extensions T satisfy that*

$$S \subseteq T \subseteq S^*. \tag{6.8}$$

Moreover, there is an one-to-one correspondence between all the unitary extensions U of the Cayley transform V and all the self-adjoint extensions T of S:

(i) If U is a unitary extension of V, i.e., $\mathcal{G}(V) \subseteq \mathcal{G}(U)$, then

$$T \overset{def}{=} i\,(U + 1_H)\,(U - 1_H)^{-1} \tag{6.9}$$

is a self-adjoint extension of S.

(ii) If conversely T is a self-adjoint extension of S, then

$$U = (T + i)\,(T - i)^{-1} \tag{6.10}$$

is a unitary extension of V.

(iii) In general, the correspondence between (i) and (ii) holds for the family of all Hermitian extensions V_T of V, and all the Hermitian extensions T of S; and the correspondence is decided by

(611) $$T = i\,(V_T + 1_H)\,(V_T - 1_H)^{-1}$$
and
(6.12) $$V_T = (T+i)\,(T-i)^{-1},$$

respectively. □

We provide several examples to illustrate our purpose.

Example 5.1. *(1) Let $H = L^2(0,\,1)$ and let*

(6.13) $$S \overset{def}{=} \tfrac{1}{i}\,\tfrac{d}{dx}$$
with
$$\mathcal{D} = dom(S) = C_0^1(0,\,1) = all\ C^1\text{-}functions\ \varphi\ on\ [0,\,1]$$
such that
$$\varphi(0) = \varphi(1) = 0.$$

Then the operator S has its deficiency indices

$$n_+ = 1 = n_-$$

and the defect spaces are

(6.14) $$\mathcal{E}_+ = \mathbb{C}\cdot e^{-x}$$
and
(6.15) $$\mathcal{E}_- = \mathbb{C}\cdot e^{x}.$$

(2) Every $\theta \in [0,\,2\pi)$ determines a unique self-adjoint extension T_θ of S as follows. Set

$$\mathcal{B}_\theta \overset{def}{=} \left\{e^{i(\theta+2\pi n)x}\ :\ n \in \mathbb{Z}\right\}.$$
Then
$$\mathcal{B}_\theta \subseteq dom\,(T_\theta)$$
and
$$T_\theta\ e^{i(\theta+2\pi\,n)x} = (\theta + 2\pi\,n)\ e^{i(\theta+2\pi\,n)x},\ for\ n \in \mathbb{Z}.$$
i.e.,
(6.16) $$spec\,(T_\theta) = \theta + 2\pi\ \mathbb{Z}\ in\ \mathbb{C},$$

where $spec(T_\theta)$ means the spectrum of T_θ.

The following proposition is the main result of this Section.

Proposition 5.2. *Let S be a Hermitian symmetric operator with its dense domain in a Hilbert space H, and assume that S has the deficiency indices* $(1, 1)$. *Then all the self-adjoint extensions of S have Cayley transform*

$$U = V \oplus W, \tag{6.17}$$

where V is the Cayley transform of S determined in (6.3), and where W has W^* W and W W^* of rank 1. *Moreover, there exists $\alpha \in \mathbb{C}$, $|\alpha| < 1$, such that*

$$W^2 = \alpha\, W. \tag{6.18}$$

Proof. The defect spaces $\mathcal{E}_+$ and $\mathcal{E}_-$ in (6.5) and (6.6) are the initial and final spaces of the partial isometries W from the previous lemma which extend the Cayley transform V of S. So,

$$W^*\ W = \mathcal{E}_+ \text{ and } W\ W^* = \mathcal{E}_-.$$

Since $\dim \mathcal{E}_{\pm 1} = 1$, there are vectors $e_{\pm 1} \in \mathcal{D}$ such that $\|e_{\pm 1}\| = 1$ and

$$\mathcal{E}_+ = \mathbb{C} \cdot e_+ \text{ and } \mathcal{E}_- = \mathbb{C} \cdot e_-.$$

It follows that

$$W = |e_- >< e_+|\,, \tag{6.19}$$

where we use the Dirac notation,

(6.20)

$$|\xi >< \eta|\, \zeta \overset{def}{=} < \eta, \zeta > \xi, \text{ for all } \zeta,\, \eta,\, \xi \in H.$$

Hence we have that

$$W^2 = < e_+, e_- > W$$

and

$$\alpha = < e_+, e_- >$$

works in (6.18).

If $|\alpha| < 1$, then, by Schuarz, it would follow that $\mathcal{E}_+ = \mathcal{E}_-$, which contradicts

$$S^*\ e_{\pm 1} = \pm i\, e_{\pm 1},$$

from (6.5) to (6.6). ∎

By the previous proposition, we can get the following corollary.

Corollary 5.3. *Let* W *be as in the previous proposition. Then* $W^n \rightarrow 0$, *as* $n \rightarrow \infty$.

Proof. By (6.18) and by Induction, we have that

$$W^{n+1} = \alpha^n \ W, \text{ for all } n \in \mathbb{N}.$$

Thus $W^n \rightarrow 0$, as $n \rightarrow \infty$. ∎

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